Application of stochastic fluids in hydro and critical limits

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Outline and Motivations

- To provide a field-theoretical justification for stochastic hydrodynamics
- To present their own equations of motion for n-particle correlators
- To reveal some unusual behavior (critical phenomena) through the framework

FLUID DYNAMICS FOR RELATIVISTIC QCD MATTER

Fluid dynamics is a universal effective field theory (EFT) of non-equilibrium many-body systems with a stable equation of state and

- Conservation of charge: $\partial_\mu J^\mu = 0$
- Conservation of energy and momentum: $\partial_\mu T^{\mu\nu} = 0$

$$
\begin{aligned}
J^\mu &= n\, u^\mu + \nu^\mu \\
T^{\mu\nu} &= \varepsilon\, u^\mu u^\nu + p\Delta^{\mu\nu} + \pi^{\mu\nu} \\
\Delta^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu \qquad \nu^\mu = -\kappa T \Delta^{\mu\nu}\partial_\nu \left(\frac{\mu}{T}\right) \\
\pi^{ij} &= -\eta\left(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\nabla\cdot\mathbf{u}\right) - \zeta\delta^{ij}\nabla\cdot\mathbf{u}
\end{aligned}
$$

The dissipation terms are described by the shear viscosity η, bulk viscosity ζ and charge conductivity κ

Dynamical Model in RHIC Beam Energy Scan

The real world is more complicated than the predictions in the first order. Additional factors must be considered, such as:

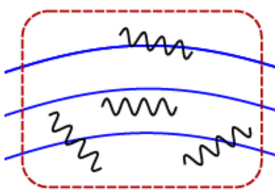
- Finite size and finite expansion rate effects
- Freeze-out, resonances, global charge conservation, and others
- Non-dissipation effects
  - The role of fluctuations is enhanced in nearly perfect fluids (long time tails)
  - Fluctuations are dominant near critical points

FLUCTUATIONS IN HYDRO

- The deterministic hydro equations do not lead to spontaneous fluctuations
- The occurrence of fluctuations is a consequence of the microscopic dynamics and must persist at the coarse-grained hydro-level

Introducing non-linear dissipation with temperature-dependent transport coefficients and random noises:

$$
\begin{aligned}
J^\mu &\rightarrow J^\mu + \theta^\mu \\
T^{\mu\nu} &\rightarrow T^{\mu\nu} + \theta^{\mu\nu}
\end{aligned}
$$



$$\langle \theta^\mu \rangle = 0 \quad \left\langle (\theta^\mu)^2 \right\rangle \sim L_J(x)\delta(x-x')(t-t')$$

$$\langle \theta^{\mu\nu} \rangle = 0 \quad \left\langle (\theta^{\mu\nu})^2 \right\rangle \sim L_T(x)\delta(x-x')(t-t')$$

Representation in MSRJD Field Theory

In terms of the slow variable (a conserved density), the free energy of the fluid:

$$\mathscr{F}[\psi] = \int d^3x \left\{ \frac{1}{2}(\vec{\nabla}\psi)^2 + \frac{r}{2}\psi(x,t)^2 + \frac{\lambda}{3!}\psi(x,t)^3 + \ldots + h(x,t)\psi(x,t) \right\}$$

The diffusion equation:

$$\partial_t \psi(x,t) = \vec{\nabla}\left\{ \kappa(\psi)\,\vec{\nabla}\left(\frac{\delta\mathscr{F}[\psi]}{\delta\psi}\right)\right\} + \theta(x,t)$$

where the Gaussian noise term $\theta(x,t)$ has a distribution

$$P[\theta] \sim \exp\left(-\frac{1}{4}\int d^3x\, dt\, \theta(x,t) L(\psi)^{-1}\theta(x,t)\right)$$

REPRESENTATION IN MSRJD FIELD THEORY, CONT.

The conductivity, $\kappa(\psi)$, is field-dependent: $\kappa(\psi) = \kappa_0 (1 + \lambda_D \psi)$

The partition function is given as: MSR, PhysRevA.8:423(1973)

$$Z = \int \mathcal{D}\psi \, P[\theta] \exp\left(-i\tilde{\psi}\,(\text{e.o.m}[\psi,\,\theta\,])\right)$$
$$= \int \mathcal{D}\psi\,\mathcal{D}\tilde{\psi}\ \exp\left(-\int d^3x\,dt\,\mathcal{L}(\psi,\tilde{\psi})\right)$$

The effective Lagrangian of this theory is:

$$\mathcal{L}(\psi,\tilde{\psi}) = \tilde{\psi}\left(\partial_t - D_0\nabla^2\right)\psi - \frac{D_0\lambda'}{2}\left(\nabla^2\tilde{\psi}\right)\psi^2 - \tilde{\psi}L(\psi)\tilde{\psi}$$

Note: $D_0 = r\kappa_0$ and $\lambda' = \lambda/r + \lambda_D$.

The noise kernel is chosen as $L(\psi) = \overleftarrow{\nabla}\left[k_B T\kappa(\psi)\right]\overrightarrow{\nabla}$

TIME REVERSAL SYMMETRY

Stochastic theories must describe the detailed balance condition:

$$\frac{P\left(\psi_1 \longrightarrow \psi_2\right)}{P\left(\psi_2 \longrightarrow \psi_1\right)} = e^{-\Delta\mathscr{F}/k_B T}$$

which is related to time-reversal symmetry:

$$\Psi(t) \longrightarrow \psi(-t)$$

$$\tilde{\Psi}(t) \longrightarrow -\left[\tilde{\psi}(-t) + \frac{\delta F}{\delta \psi}\right]$$

$$\mathscr{L} \longrightarrow \mathscr{L} + \frac{d}{dt}F$$

Janssen, ZPhyB.23:377(1976)

The form and magnitude of $\kappa(\psi)$ is determined by TSR

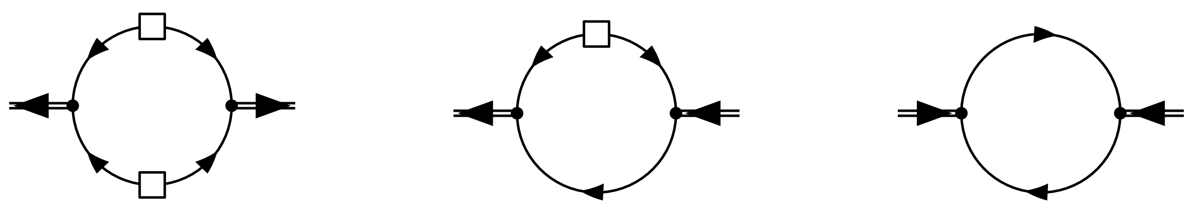
1PI Effective Action

Consider the generating functional with local source J, $\tilde{J}$:

$$W[J,\tilde{J}] = -\ln \int \mathcal{D}\psi \mathcal{D}\tilde{\psi}\, e^{-\int dt\, d^3x \left\{\mathcal{L} + J\psi + \tilde{J}\tilde{\psi}\right\}}$$

Performing a Legendre transform to the 1PI effective action via background field method with $\psi = \Psi + \delta\psi$:

$$\Gamma[\Psi,\tilde{\Psi}] = W[J,\tilde{J}] - \int dt\, d^3x \left(J\Psi + \tilde{J}\tilde{\Psi}\right)$$



Taking the derivative of the 1PI effective action w.r.t. the classical field Ψ yields the e.o.m. encoded the fluctuation effects:

$$(\partial_t - D\nabla^2)\Psi - \frac{\kappa\lambda_3^2}{2}\nabla^2\Psi^2 + \int d^3x'\, dt'\, \Psi(x',t')\Sigma(x,t;x',t') = 0$$

DOUBLE LEGENDRE TRANSFORMATION

☞ nPI effective action $\Longrightarrow$ e.o.m. for n-point functions

✓ Couple a bi-local source $\frac{1}{2}\psi_a K_{ab}\psi_b$ to the system 🔗 Cornwall, Jackiw and Tomboulis, PhysRevD.10:2428 (1974)

✓ Plug in the 1-loop 1PI effective action

✓ Sum beyond 1-loop terms

✓ Apply the stationary conditions:

$$\frac{\delta W}{\delta J_a} = \langle\psi_a\rangle = \Psi_a\,, \qquad \frac{\delta W}{\delta K_{ab}} = \frac{1}{2}\langle\psi_a\psi_b\rangle = \frac{1}{2}[\Psi_a\Psi_b + G_{ab}]$$

$$\Gamma[\Psi_a, G_{ab}] = W[J_a, K_{ab}] - J_A\Psi_A - \frac{1}{2}K_{AB}[\Psi_A\Psi_B + G_{AB}]$$

2PI Effective Action

The 2PI effective action is given by:

$$\Gamma[\Psi_a, G_{ab}] = S[\Psi_a] + \frac{1}{2}\frac{\delta^2 S}{\delta\Psi_A \delta\Psi_B} G_{AB} - \frac{1}{2}\mathrm{Tr}\left[\log(G)\right] + \Gamma_F[\Psi_a, G_{ab}]$$

The higher order fluctuations are:

$$\begin{aligned}\exp(-\Gamma_F[\Psi_a, G_{ab}]) \quad = \quad & \frac{1}{\sqrt{\det(G)}}\int D(\delta\psi_a)\,\exp\left\{ -\frac{1}{2}\delta\psi_A (G^{-1})_{AB}\delta\psi_B \right. \\ & \left. - \left[S_3[\Psi_a, \delta\psi_a] - \bar{J}_A\delta\psi_A - \bar{K}_{AB}(\delta\psi_A\delta\psi_B - G_{AB})\right] \right\}\end{aligned}$$

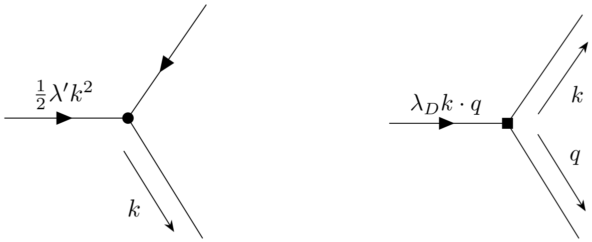
with

$$\bar{J}_a = \frac{1}{2}\frac{\delta^3 S}{\delta\Psi_a\delta\Psi_B\delta\Psi_C} G_{BC} + \frac{\delta\Gamma_F}{\delta\Psi_a}, \quad \bar{K}_{ab} = \frac{\delta\Gamma_F}{\delta G_{ab}}$$

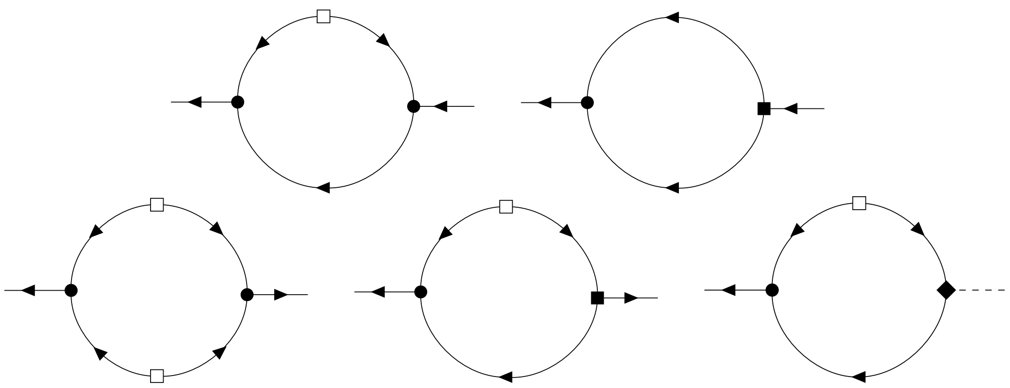
Simpler Example of Model B

- Linearized propagator:

- Vertex and new vertices:

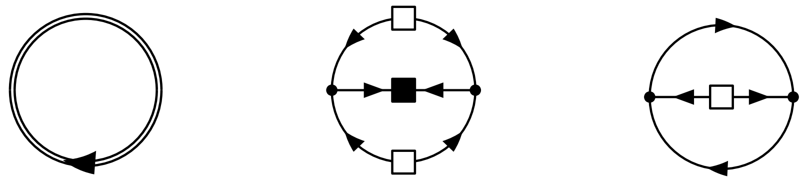


- Loop contributions:



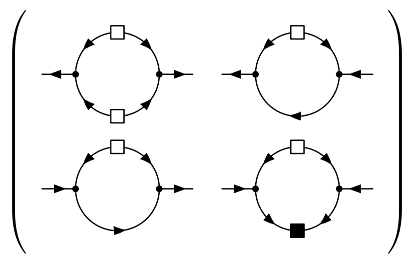
DSE in Mixed Representation

The loop diagrams generated by Γ_F use the full propagator G_{ab}:



Taking the derivative w.r.t G, obtain the DS equation ($\tilde{\psi}$, ψ = 1, 2):

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$

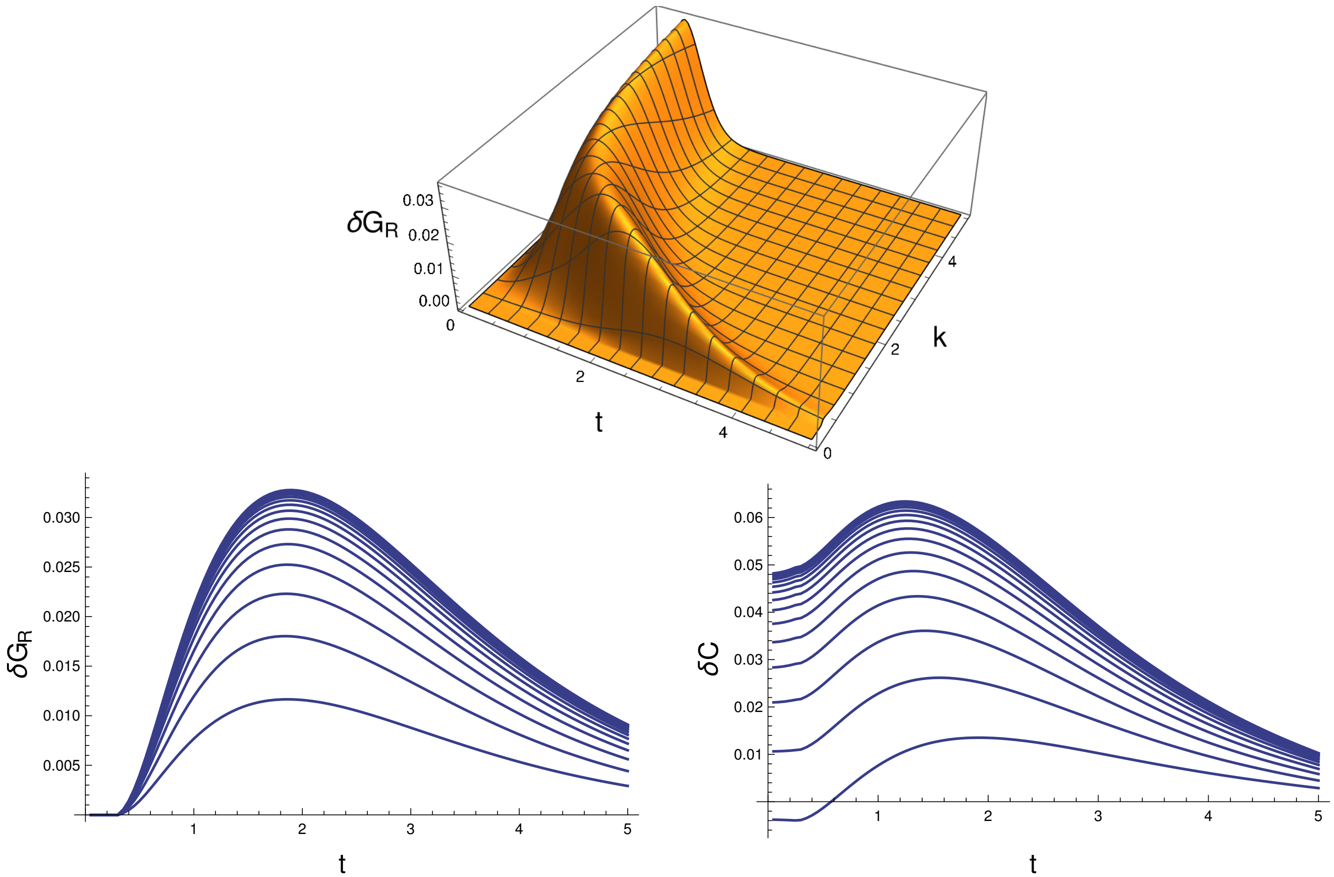


In time-momentum mixed representation

$$\Sigma(t, k^2) = (\kappa\lambda_3)^2 \int d^3k' \, k^2 (k + k')^2 \, C(t, k') \, G_R(t, k + k') \, ,$$

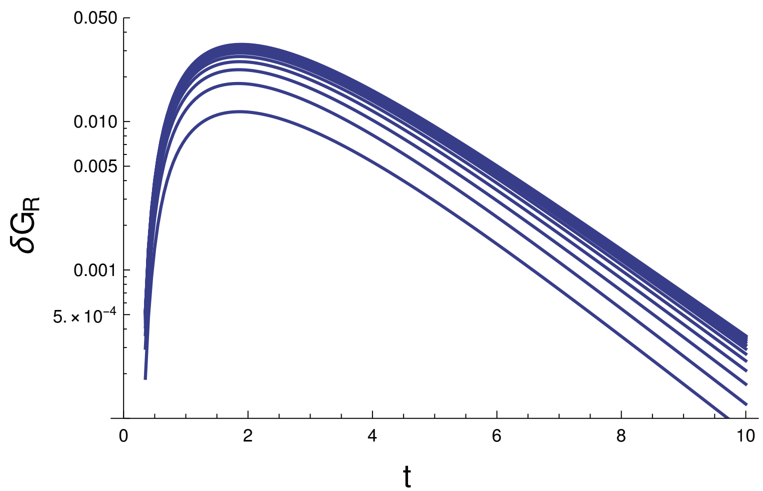
$$\delta D(t, k^2) = \frac{(\kappa\lambda_3)^2}{2} \int d^3k' \, k^4 C(t, k') \, C(t, k + k')$$

NUMERICAL SIMULATIONS IN HYDRO LIMIT

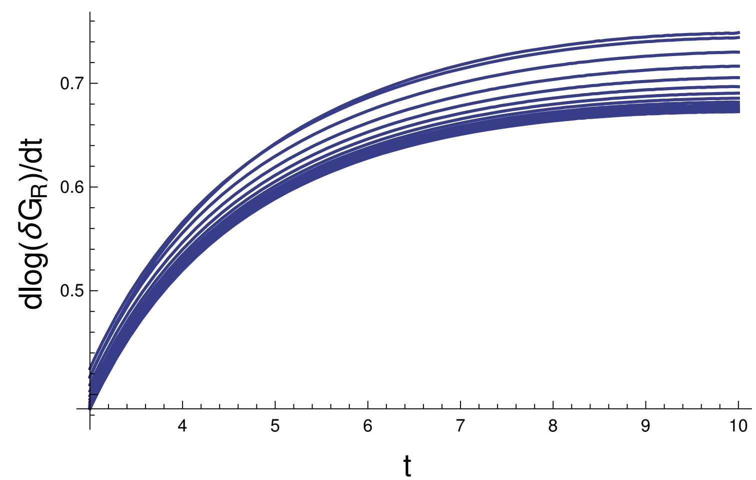


3D curve of $\delta G(t, k)$ and the iterative solutions of DSE

LONG-TIME BEHAVIOR



A logarithmic plot of the loop corrections to the retarded $\delta G_R(k, t)$



The logarithmic derivative of $\delta G_R(k, t)$ w.r.t t

The long-time behavior of the diffusion cascade is conjectured to be $\sim n! \exp(-Dk^2 t/n)$ because of the n-loop terms (shown but not reached).

Delacretaz, SciPostPhys.9:034(2020)

Mode Coupling Theory (MCT)

- For non-critical fluids, use the gradient expansion method where $k\xi \ll 1$
- For critical fluids, their behaviors are characterized by the transport coefficients in the MCT (= Poisson bracket terms + the critical transport coeffcients)

By applying an uncontrolled approximation within the MCT, the well known retarded function $G^{-1}(\omega, k) = i\omega - \Gamma_k$ of the diffusion mode is modified to:

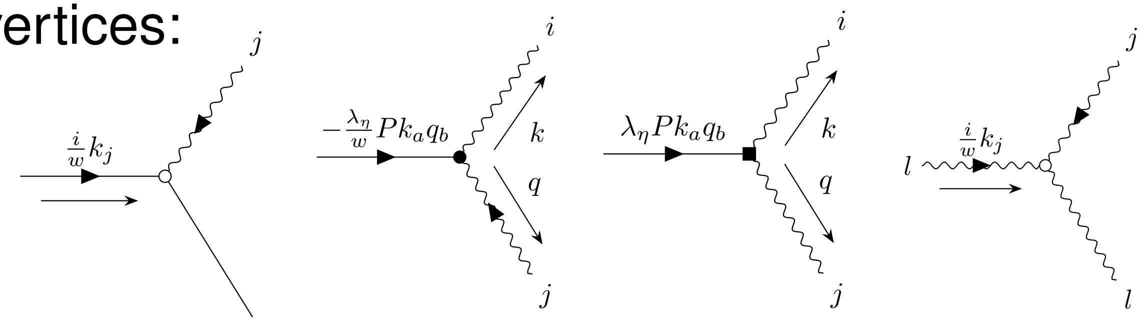
$$\Gamma_k = \frac{T}{6\pi\eta_0\xi^3}K(k\xi) \quad \text{with} \quad K(k\xi = x) = \frac{3}{4}\left[1 + x^2 + (x^3 - x^{-1})\arctan(x)\right]$$

η_0 is the bare shear viscosity. Kawasaki, AnnPhys.61:1(1970); JC and T. Schaefer, work-in-process

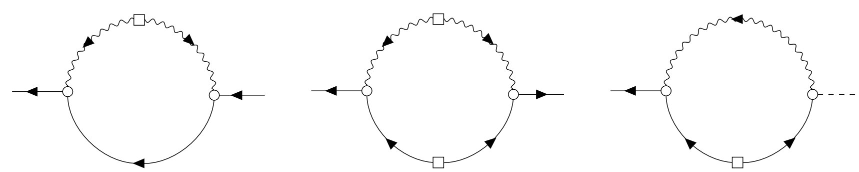
Model H

- Linearized propagator:

- Vertices and new vertices:

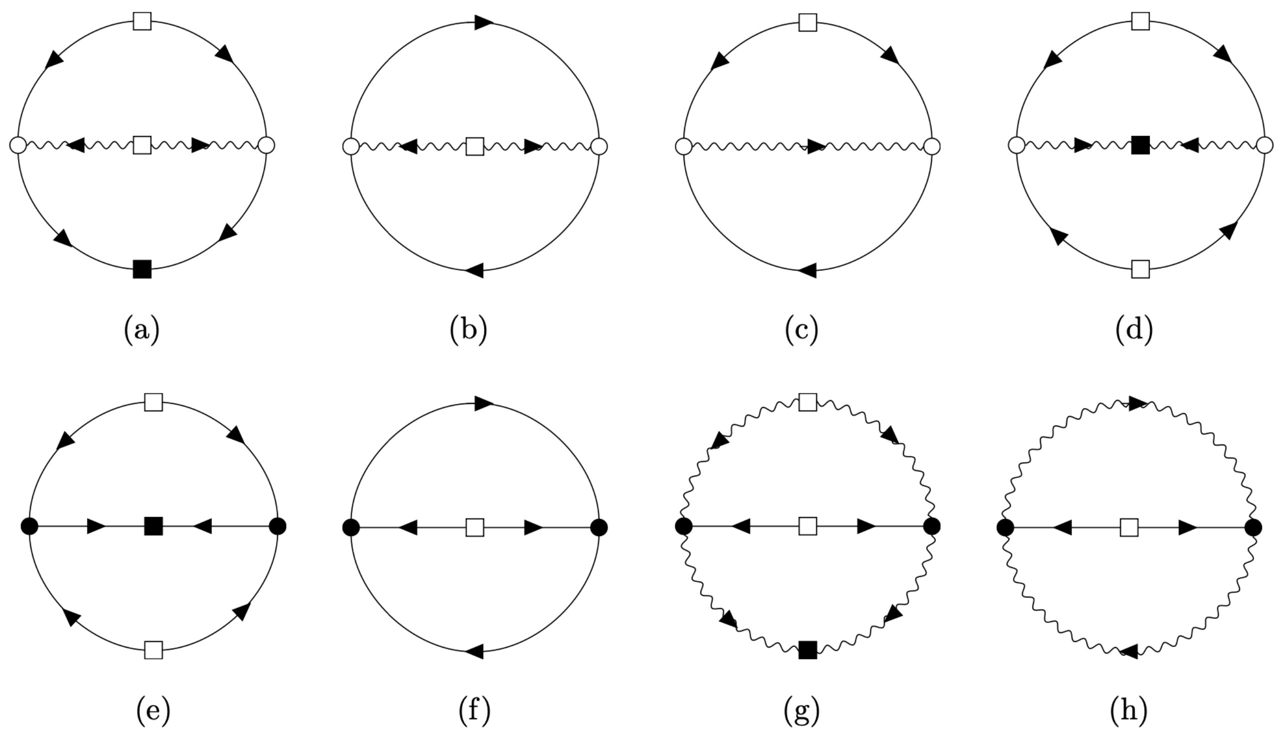


- Mode-coupling loop contributions:



☞ the multiplicative noise contribution to the tails is subleading compared to the contributions induced by mode couplings in hydro limit

2PI Effective Action in Model H



Above: The traditional contribution, which originates from the vertex of the Poisson bracket, is illustrated within the MCT
Below: Additional contributions are derived from the newer vertex

Scaling Forms of the Transport Coefficients

The modified critical transport coefficients:

$$\begin{aligned} D \to D^c(\omega, k, \xi) &= D\,(k\xi)^{x_D} F_D\,(\omega\xi^z, k\xi) \\ \kappa \to \kappa^c(\omega, k, \xi) &= \kappa\,(k\xi)^{x_\kappa} F_\kappa\,(\omega\xi^z, k\xi) \\ \eta \to \eta^c(\omega, k, \xi) &= \eta\,(k\xi)^{x_\eta} F_\eta\,(\omega\xi^z, k\xi) \\ \gamma \to \gamma^c(\omega, k, \xi) &= \gamma\,(k\xi)^{x_\gamma} F_\gamma\,(\omega\xi^z, k\xi) \end{aligned}$$

- Contrary to the hydrodynamic limit, where $D = \kappa m^2$, $\eta = \gamma w$ and w is enthalpy
- The relaxation frequency scales as $\omega \sim k^z$
- The dynamical exponent z is determined as $z = 4 - \eta + x_D$ for the diffusion mode in the regime where $k \gg \xi^{-1}$

CRITICAL SELF-ENERGY FUNCTIONS

The Ornstein-Zernike form is utilized, expressed as $\chi^{-1}(x) = g(x) = 1 + x^2$, with the static critical exponent set to $\eta = 0$. And then,

$$\begin{aligned}
\Sigma^c_{12}(0,x) &= D\,x^{2+x_D} g(x) F_D(0,x)\,, \\
\Sigma^c_{11}(0,x) &= \kappa\,x^{2+x_\kappa} F_\kappa(0,x)\,, \\
\Delta^c_{12}(0,x) &= \gamma\,x^{2+x_\gamma} F_\gamma(0,x)\,, \\
\Delta^c_{11}(0,x) &= \eta\,x^{2+x_\eta} F_\eta(0,x)\,,
\end{aligned}$$

where

$$F_i(s=0, x\to\infty) = F_i^\infty = \text{constant}$$

and

$$F_i(s=0, x\to 0) = F_i^0\,x^{-x_i}$$

with $i = D, \kappa, \gamma, \eta$.

UV Finite Self-Consistent Equations

Re-scale the frequency and the momentum as $(s, r) = (\omega\xi^z, \omega'\xi^z)$, $(x, y) = (k\xi, k'\xi)$ and $(\Sigma^c, \Delta^c) = x^2(\Sigma^c, \Delta^c)$, the self-energies are:

$$
\begin{aligned}
\Sigma^c_{12}(s,x) \quad = \quad & \xi^{-3}x^2 \int_{r,y} \left\{ \frac{\xi^{z-2}\Sigma^c_{11}(r_-,y_-)}{r_-^2 + |\xi^{z-4}\Sigma^c_{12}(r_-,y_-)|^2} \frac{(\kappa\lambda_3)^2 y_+^2}{i\,r_+ + \xi^{z-4}\Sigma^c_{12}(-r_+,y_+)} \right. \\
& - \frac{\xi^{z-2}y^2\Delta^c_{11}(r_-,y_-)}{r_-^2 + |\xi^{z-2}\Delta^c_{12}(r_-,y_-)|^2} \frac{\xi^2}{w^2 y_-^2} \frac{1-(\hat{x}\cdot\hat{y})^2}{i\,r_+ + \xi^{z-4}\Sigma^c_{12}(-r_+,y_+)} \\
& \left. - \frac{\xi^{z-2}y^2\Sigma^c_{11}(r_-,y_-)}{r_-^2 + |\xi^{z-4}\Sigma^c_{12}(r_-,y_-)|^2} \frac{1-(\hat{x}\cdot\hat{y})^2}{w y_+^2} \frac{x^2 - y_-^2}{i\,r_+ + \xi^{z-2}\Delta^c_{12}(-r_+,y_+)} \right\} ,
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^c_{11}(s,x) \quad = \quad & \xi^{-5}x^2 \int_{r,y} \left\{ \frac{\xi^{z-2}\Sigma^c_{11}(r_-,y_-)}{r_-^2 + |\xi^{z-4}\Sigma^c_{12}(r_-,y_-)|^2} \frac{(\kappa\lambda_3)^2}{2} \frac{x^2\,\xi^{z-2}\Sigma^c_{11}(r_+,y_+)}{r_+^2 + |\xi^{z-4}\Sigma^c_{12}(r_+,y_+)|^2} \right. \\
& \left. + \frac{\xi^{z-2}y^2\Delta^c_{11}(r_-,y_-)}{r_-^2 + |\xi^{z-2}\Delta^c_{12}(r_-,y_-)|^2} \frac{\xi^2}{w^2} \frac{1-(\hat{x}\cdot\hat{y})^2}{y_-^2} \frac{\xi^{z-2}\Sigma^c_{11}(r_+,y_+)}{r_+^2 + |\xi^{z-4}\Sigma^c_{12}(r_+,y_+)|^2} \right\} ,
\end{aligned}
$$

$$\Delta_{12}^{c}(s,x) = \xi^{-5}x^2 \int_{r,y} \left\{ \frac{(\gamma\lambda_\eta)^2\,\xi^{z-2}\Sigma_{11}^{c}(r_-,y_-)}{r_-^2 + \left|\xi^{z-4}\Sigma_{12}^{c}(r_-,y_-)\right|^2} \frac{\mathscr{P}_t(x,y)(\hat{x}\cdot\vec{y}_+)^2}{i\,r_+ + \xi^{z-2}\Delta_{12}^{c}(-r_+,y_+)} - \frac{\xi^{z-2}\Sigma_{11}^{c}(r_-,y_-)}{r_-^2 + \left|\xi^{z-4}\Sigma_{12}^{c}(r_-,y_-)\right|^2} \frac{1-(\hat{x}\cdot\hat{y})^2}{wx} \frac{2\,\hat{x}\cdot\hat{y}\,y^3}{i\,r_+ + \xi^{z-4}\Sigma_{12}^{c}(-r_+,y_+)} \right\},$$

$$\Delta_{11}^{c}(s,x) = \xi^{-5}x^2 \int_{r,y} \left\{ \frac{(\gamma\lambda_\eta)^2\,\xi^{z-2}\Sigma_{11}^{c}(r_-,y_-)}{r_-^2 + \left|\xi^{z-4}\Sigma_{12}^{c}(r_-,y_-)\right|^2} \frac{\mathscr{P}_t(x,y)(\hat{x}\cdot\vec{y}_+)^2\xi^{z-2}\Delta_{11}^{c}(r_+,y_+)}{r_+^2 + \left|\xi^{z-2}\Delta_{12}^{c}(r_+,y_+)\right|^2} + \frac{\xi^{z-2}\Sigma_{11}^{c}(r_-,y_-)}{r_-^2 + \left|\xi^{z-4}\Sigma_{12}^{c}(r_-,y_-)\right|^2} \frac{1-(\hat{x}\cdot\hat{y})^2}{\xi^2 x} \frac{2\,\hat{x}\cdot\hat{y}\,y^3\,y_+^2\,\xi^{z-2}\Sigma_{11}^{c}(r_+,y_+)}{r_+^2 + \left|\xi^{z-4}\Sigma_{12}^{c}(r_+,y_+)\right|^2} \right\},$$

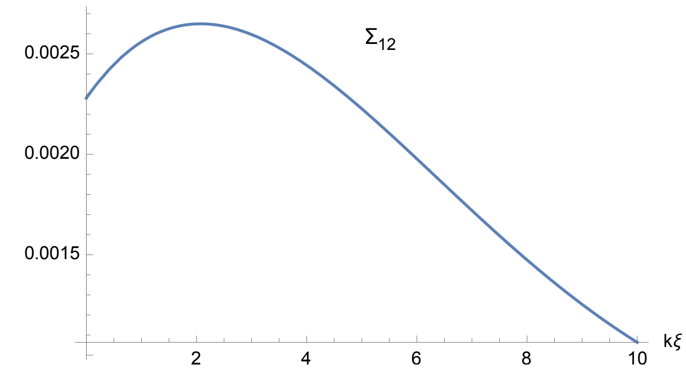
where $\mathscr{P}_t(x,y) = 1 + (x^2 + \vec{x}\cdot\vec{y})^2\, x^{-2}(\vec{x}+\vec{y})^{-2}$

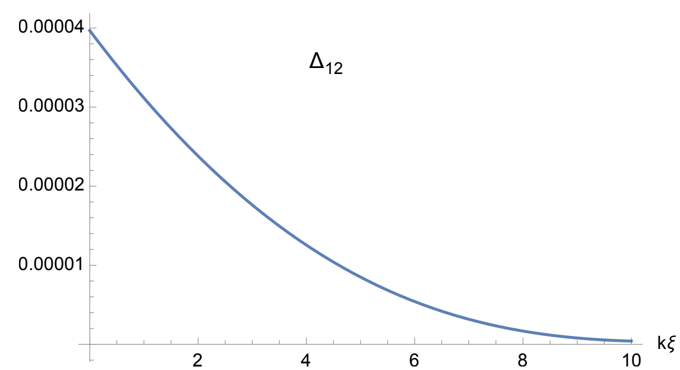
$$\int_{r,y} = \int_{-\omega_\Lambda}^{\omega_\Lambda} dr \int_0^{\Lambda} \frac{y^2}{(2\pi)^d}\, dy \int_0^{\pi} \sin\theta\, d\theta$$

Note: Although the integration is UV finite, the UV cutoffs ω_Λ and Λ are introduced, both of which are proportional to the microscopic scale of ξ_0^{-1}.

Numerical Simulations in Critical Limit (prelimary)

z	$\xi_0\,(\sim 1/\Lambda)$	$a_0\,(\sim \omega_\Lambda \xi_0^3)$
3.092	0.117	0.317
3.080	0.125	
3.073	0.133	
3.077	0.142	
3.083	0.133	0.399
3.075		0.352
3.073		0.317
3.077		0.282





correlation length $\xi = 2$