Exact solution of hydrodynamic and Boltzmann equations in longitudinally expanding systems

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Shile Chen(陈诗乐) and SS, 2311.09575

SS, S. Jeon, and C. Gale, Phys.Rev.C(Letter) 105 (2022) 2, L021902

Found (1+1)d analytical solutions

to rel. ideal hydro eqs. w/ simple EoS:

$$\mathscr{D}_\mu T^{\mu\nu} \equiv \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\rho\mu} T^{\rho\nu} + \Gamma^\nu{}_{\rho\mu} T^{\rho\mu} = 0\,,$$
$$T^{\mu\nu} = (\varepsilon + P)\, u^\mu u^\nu - P\, g^{\mu\nu}\,,$$
$$P = c_s^2 \varepsilon\,.$$

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$$\frac{\varepsilon}{\varepsilon_0} = \left(\frac{t_0}{\tau_0} + \frac{a\,\tau}{\tau_0} e^{\eta-\eta_0}\right)^{\frac{1-c_s^4}{4c_s^2}\frac{1}{a^2} - \frac{(1+c_s^2)^2}{4c_s^2}} \left(\frac{t_0}{\tau_0} + \frac{\tau}{a\,\tau_0} e^{\eta_0-\eta}\right)^{\frac{1-c_s^4}{4c_s^2}a^2 - \frac{(1+c_s^2)^2}{4c_s^2}},$$

$$u^\tau = \frac{1}{2}\left(\sqrt{\frac{t_0 e^{\eta_0-\eta} + \tau\,a}{t_0 e^{\eta-\eta_0} + \tau/a}} + \sqrt{\frac{t_0 e^{\eta-\eta_0} + \tau/a}{t_0 e^{\eta_0-\eta} + \tau\,a}}\right),$$

$$u^\eta = \frac{1}{2\tau}\left(\sqrt{\frac{t_0 e^{\eta_0-\eta} + \tau\,a}{t_0 e^{\eta-\eta_0} + \tau/a}} - \sqrt{\frac{t_0 e^{\eta-\eta_0} + \tau/a}{t_0 e^{\eta_0-\eta} + \tau\,a}}\right),$$

$$u^x = u^y = 0\,.$$

variables:

- τ: proper time
- η: spatial rapidity

parameters (constants):

- c_s: speed of sound
- η_0: Lorentz boost along $\hat{z}$
- τ_0: scaling of time
- ε_0: scaling of energy density
- a: [dimensionless]
- t_0: [time unit]

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$$T^{\mu\nu} = (\varepsilon + P)\, u^\mu u^\nu - P\, g^{\mu\nu}\,,$$

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proof

Coordinates, Metric and Connections

In[1]:=
```
x1 = {τ Cosh[η], τ Sinh[η], x, y};
```

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

```
x2 = {τ, η, x, y};
```

$$\text{gDown} = \text{Table}\left[\text{FullSimplify}\left[\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4} H[[\alpha, \beta]]\, \partial_{x2[[\mu]]}\, x1[[\alpha]]\, \partial_{x2[[\nu]]}\, x1[[\beta]]\right], \{\mu, 4\}, \{\nu, 4\}\right];$$

```
gUp = Inverse[gDown];
```

$$\Gamma = \text{Table}\left[\text{FullSimplify}\left[\frac{1}{2}\sum_{\alpha=1}^{4} \text{gUp}[[\mu, \alpha]] \left(\partial_{x2[[\rho]]}\, \text{gDown}[[\nu, \alpha]] + \partial_{x2[[\nu]]}\, \text{gDown}[[\rho, \alpha]] - \partial_{x2[[\alpha]]}\, \text{gDown}[[\nu, \rho]]\right)\right], \{\mu, 4\}, \{\nu, 4\}, \{\rho, 4\}\right];$$

```
Print["gμν=", MatrixForm[gDown], ",\t gμν=",
 MatrixForm[gUp], ",\nΓ=", Γ]
```

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tau^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\tau^2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\Gamma = \Big\{\{\{0,0,0,0\},\{0,\tau,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}, \left\{\left\{0,\tfrac{1}{\tau},0,0\right\},\left\{\tfrac{1}{\tau},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\}, \{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}, \{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\Big\}$$

Check the Solution

In[8]:=

$$p = p0\left(\frac{t0}{\tau0} + \frac{a\,\tau}{\tau0}\, e^{\eta-\eta0}\right)^{\frac{1-c^4}{4c^2}\frac{1}{a^2} - \frac{(1+c^2)^2}{4c^2}} \left(\frac{t0}{\tau0} + \frac{\tau}{a\,\tau0}\, e^{\eta0-\eta}\right)^{\frac{1-c^4}{4c^2}a^2 - \frac{(1+c^2)^2}{4c^2}};$$

$$u\tau = \frac{1}{2}\left(\sqrt{\frac{t0\, e^{\eta0-\eta} + \tau\, a}{t0\, e^{\eta-\eta0} + \tau / a}} + \sqrt{\frac{t0\, e^{\eta-\eta0} + \tau / a}{t0\, e^{\eta0-\eta} + \tau\, a}}\right);$$

$$u\eta = \frac{1}{2\tau}\left(\sqrt{\frac{t0\, e^{\eta0-\eta} + \tau\, a}{t0\, e^{\eta-\eta0} + \tau / a}} - \sqrt{\frac{t0\, e^{\eta-\eta0} + \tau / a}{t0\, e^{\eta0-\eta} + \tau\, a}}\right);$$

```
u = {uτ, uη, 0, 0};
```

$$\text{TUp} = \text{Table}\left[\left(1 + c^{-2}\right) p\, u[[\mu]]\, u[[\nu]] - p\, \text{gUp}[[\mu, \nu]], \{\mu, 4\}, \{\nu, 4\}\right];$$

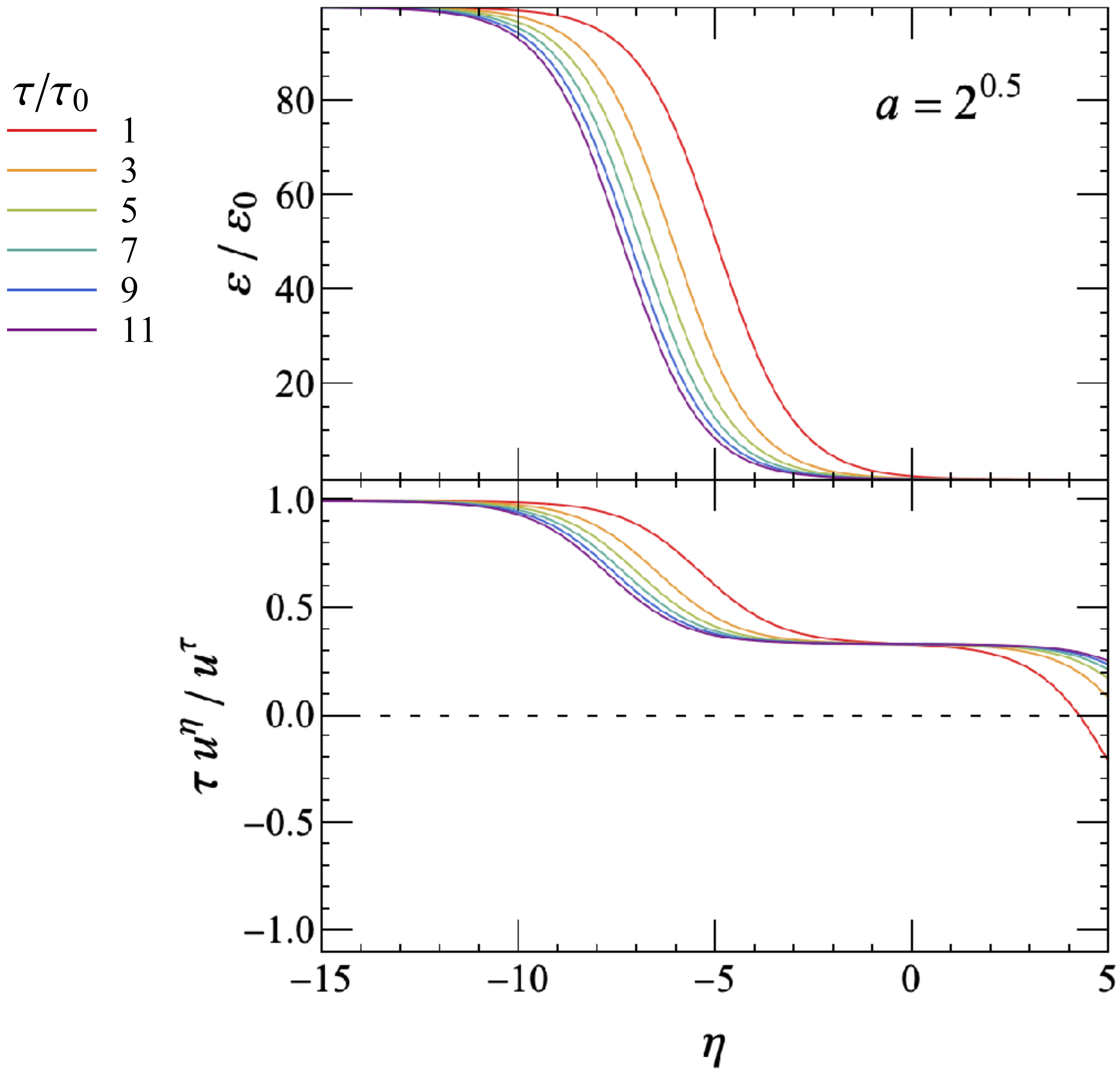
$$\text{Eq} = \text{Table}\Big[\text{FullSimplify}\Big[\text{Sum}\Big[\partial_{x2[[\nu]]}\, \text{TUp}[[\mu, \nu]] + \sum_{\alpha=1}^{4} \left(\Gamma[[\mu, \alpha, \nu]]\, \text{TUp}[[\nu, \alpha]] + \Gamma[[\nu, \alpha, \nu]]\, \text{TUp}[[\mu, \alpha]]\right), \{\nu, 1, 4\}\Big],$$
$$\text{Assumptions} \to \{p0 > 0, a > 0, \tau > 0, \eta \in \text{Reals}, \eta0 \in \text{Reals}, \tau0 > 0, t0 > 0\}\Big], \{\mu, 4\}\Big]$$

Out[13]= {0, 0, 0, 0}

parameter dependence

$c_s = 1/\sqrt{3}$ (speed of sound)
$\eta_0 = 0$ (Lorentz boost along $\hat{z}$)

$t_0 = 0.01\tau_0$



$$\sqrt{\frac{1-c_s^2}{1+c_s^2}} \le a \le \sqrt{\frac{1+c_s^2}{1-c_s^2}}$$

controls the degree of asymmetry

$a \to 1/a$: parity reflection

$$\eta \to -\eta, \quad u^\eta \to -u^\eta$$

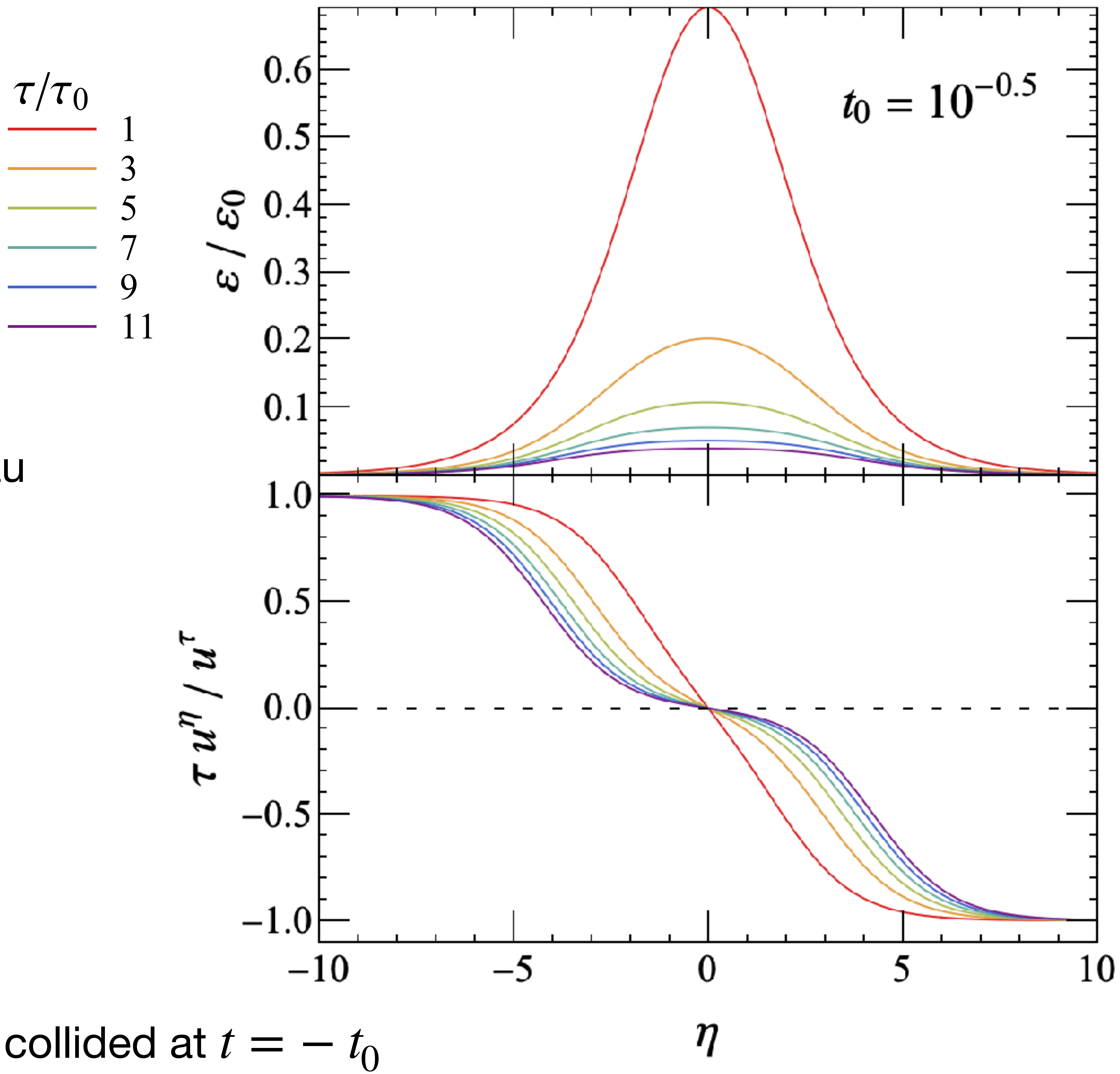
parameter dependence

$c_s = 1/\sqrt{3}$ (speed of sound)
$\eta_0 = 0$ (Lorentz boost along $\hat{z}$)

$a = 1$

$0 \leq t_0$ controls the width of the plateau

$$\frac{\varepsilon}{\varepsilon_0} = \left(\frac{\tau^2 + 2t_0\tau\cosh\eta + t_0^2}{\tau_0^2}\right)^{-\frac{1+c_s^2}{2}}$$
$$= \left(\frac{(t+t_0)^2 - z^2}{\tau_0^2}\right)^{-\frac{1+c_s^2}{2}}$$



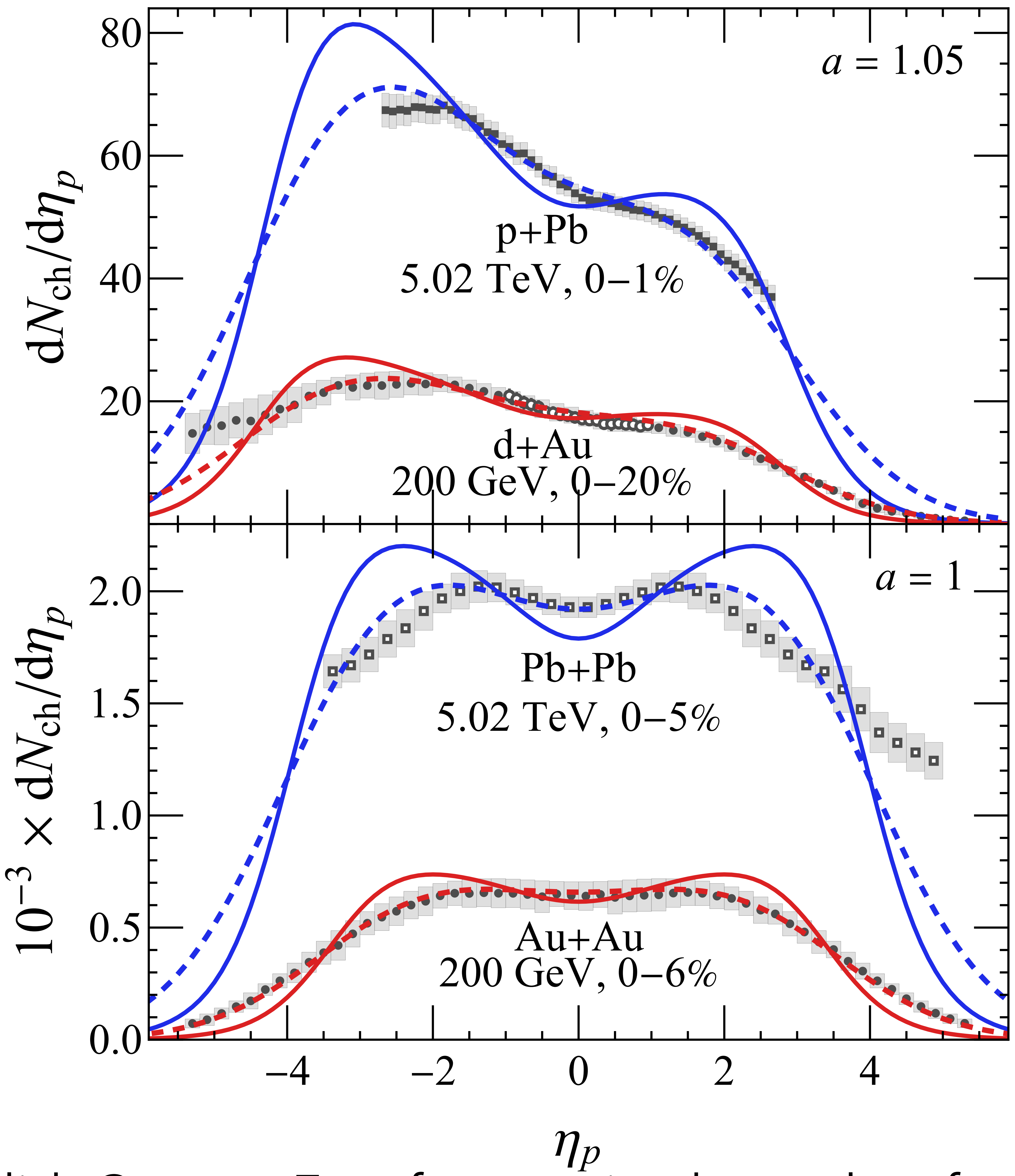
t_0: overlap time -- hydro starts at $t = 0$, collided at $t = -t_0$

phenomenology?

solid: *Qualitatively* consistent w/ exp.

Plateau *Asymmetric* in pA

Quantitatively agreement if smeared



solid: Cooper-Frye from an isothermal surface

dash: hadron cascade (gaussian convolution)

more insights? change of frame

$$x^\mu = \{\tau, \eta, x, y\}$$

$$\hat{x}^\mu = \{\hat{x}^0, \hat{x}^1, x, y\}$$

$$\frac{T_{\text{ideal}}}{T_i} = \left(\frac{t_0}{\tau_i} + \frac{a\,\tau e^{\eta}}{\tau_i}\right)^{\frac{1-c_s^2}{4}\frac{1}{a^2} - \frac{1+c_s^2}{4}} \times \left(\frac{t_0}{\tau_i} + \frac{\tau e^{-\eta}}{a\,\tau_i}\right)^{\frac{1-c_s^2}{4}a^2 - \frac{1+c_s^2}{4}},$$

$$u^\tau = \frac{1}{2}\left(\sqrt{\frac{t_0 e^{-\eta} + \tau\,a}{t_0 e^{\eta} + \tau/a}} + \sqrt{\frac{t_0 e^{\eta} + \tau/a}{t_0 e^{-\eta} + \tau\,a}}\right),$$

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more insights? change of frame

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$$\hat{x}^\mu = \{\hat{x}^0, \hat{x}^1, x, y\}$$

$$\hat{x}^0 = \frac{2a\,\tau_i}{1+a^2}\left(\left(\frac{t_0 + a\,\tau e^\eta}{\tau_i}\right)^{\frac{1}{a}}\left(\frac{t_0}{\tau_i} + \frac{\tau e^{-\eta}}{a\,\tau_i}\right)^a\right)^{\frac{1+a^2}{4a}},$$

$$\hat{x}^1 = \frac{1+a^2}{4a}\ln\left(\left(\frac{t_0 + a\,\tau e^\eta}{\tau_i}\right)^{\frac{1}{a}} \Big/ \left(\frac{t_0}{\tau_i} + \frac{\tau e^{-\eta}}{a\,\tau_i}\right)^a\right).$$

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$$\hat{T}_{\rm ideal} = \sqrt{\hat{g}^{00}}\, T_i \left(\frac{\tau_i}{\hat{x}^0}\right)^{c_s^2},$$

$$\hat{u}^0 = \sqrt{\hat{g}^{00}}\,, \quad \hat{u}^1 = \hat{u}^x = \hat{u}^y = 0\,.$$

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$$\hat{g}^{\mu\nu} = {\rm diag}\left[e^{2\frac{1-a^2}{1+a^2}\hat{x}^1}, -(\hat{x}^0)^{-2}e^{2\frac{1-a^2}{1+a^2}\hat{x}^1}, -1, -1\right],$$

$$\hat{T}_{\rm ideal} = \sqrt{\hat{g}^{00}}\, T_i \left(\frac{\tau_i}{\hat{x}^0}\right)^{c_s^2},$$

$$\hat{u}^0 = \sqrt{\hat{g}^{00}}, \quad \hat{u}^1 = \hat{u}^x = \hat{u}^y = 0.$$

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more insights? change of frame

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more insights? change of frame

"ture" 1+1D system:

—— no transverse coordinates

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Milne metric x global re-scaling

$$\hat{g}^{\mu\nu} = \mathrm{diag}\left[1, -(\hat{x}^0)^{-2}\right] \times e^{2\frac{1-a^2}{1+a^2}\hat{x}^1},$$

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$$\hat{u}^0 = \sqrt{\hat{g}^{00}}, \quad \hat{u}^1 \cancel{= \hat{u}^x = \hat{u}^y} = 0.$$

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"ture" 1+1D system:

—— no transverse coordinates

"false" 1+1D system:

—— with transverse coordinates,

but homogeneous…

$$\hat{x}^\mu = \{\hat{x}^0, \hat{x}^1, x, y\}$$

~~Milne metric x global re-scaling~~

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more insights? change of frame

"ture" 1+1D system:

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"false" 1+1D system:

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but homogeneous…

better properties in "true" 1+1D

$$\hat{x}^\mu = \{\hat{x}^0, \hat{x}^1, x, y\}$$

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Boltzmann eq. w/ RTA

$$\left(\hat{p}^{\mu}\hat{\partial}_{\mu}+\hat{\Gamma}^{\rho}{}_{\mu\nu}\hat{p}^{\mu}\hat{p}_{\rho}\frac{\partial}{\partial\hat{p}_{\nu}}\right)f(\hat{x}^{\alpha},\hat{p}_{\beta})=\mathscr{C}[f]\,,$$

$$\mathscr{C}[f]=\frac{\hat{p}_{\mu}\hat{u}^{\mu}(x)}{\tau_{r}(\hat{x})}\Big(f(\hat{x}^{\alpha},\hat{p}_{\beta})-f_{eq}(\hat{x}^{\alpha},\hat{p}_{\beta})\Big)\,,$$

$$\hat{x}^{\mu}=\{\hat{x}^{0},\hat{x}^{1},x,y\}$$

$$\hat{g}^{\mu\nu}=\mathrm{diag}\left[e^{2\frac{1-a^{2}}{1+a^{2}}\hat{x}^{1}},-(\hat{x}^{0})^{-2}e^{2\frac{1-a^{2}}{1+a^{2}}\hat{x}^{1}},-1,-1\right],$$

Boltzmann eq. w/ RTA

$$\left(\hat{p}^\mu \hat{\partial}_\mu + \hat{\Gamma}^\rho{}_{\mu\nu} \hat{p}^\mu \hat{p}_\rho \frac{\partial}{\partial \hat{p}_\nu}\right) f(\hat{x}^\alpha, \hat{p}_\beta) = \mathscr{C}[f],$$

$$\mathscr{C}[f] = \frac{\hat{p}_\mu \hat{u}^\mu(x)}{\tau_r(\hat{x})} \left(f(\hat{x}^\alpha, \hat{p}_\beta) - f_{eq}(\hat{x}^\alpha, \hat{p}_\beta) \right),$$

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$$\hat{g}^{\mu\nu} = \mathrm{diag}\left[e^{2\frac{1-a^2}{1+a^2}\hat{x}^1}, -(\hat{x}^0)^{-2} e^{2\frac{1-a^2}{1+a^2}\hat{x}^1}, -1, -1 \right],$$

formal analytical solutions:

- Milne coordinates [Florkowski, Ryblewski, Strickland PRC 88 024903]
- Gubser coordinates [Denicol, Heinz, Martinez, Noronha, Strickland PRL 113 202301]
- our new coordinates?

Boltzmann eq. w/ RTA: solution

$$\left(\hat{p}^\mu \hat{\partial}_\mu + \hat{\Gamma}^\rho{}_{\mu\nu} \hat{p}^\mu \hat{p}_\rho \frac{\partial}{\partial \hat{p}_\nu}\right) f(\hat{x}^\alpha, \hat{p}_\beta) = \mathscr{C}[f]\,,$$

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“ture” 1+1D system

“false” 1+1D system

Boltzmann eq. w/ RTA: solution

$$\left(\hat{p}^\mu \hat{\partial}_\mu + \hat{\Gamma}^\rho{}_{\mu\nu}\hat{p}^\mu \hat{p}_\rho \frac{\partial}{\partial \hat{p}_\nu}\right) f(\hat{x}^\alpha, \hat{p}_\beta) = \mathscr{C}[f]\,,$$

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$$\hat{x}^\mu = \{\hat{x}^0, \hat{x}^1, \sout{x, y}\}$$

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"ture" 1+1D system ✔

"false" 1+1D system

$$\hat{T}^2_{\text{eff}}(\hat{x}^0) = \frac{(\hat{x}^0)^2}{(\hat{x}^0_i)^2} e^{-\frac{1}{5\bar{\eta}}\int_{\hat{x}^0_i}^{\hat{x}^0} \hat{T}_{\text{eff}}(x')dx'} \hat{T}^2_0$$

$$+ \frac{1}{5\bar{\eta}} \int_{\hat{x}^0_i}^{\hat{x}^0} dx' \frac{(\hat{x}^0)^2}{(\hat{x}')^2} e^{-\frac{1}{5\bar{\eta}}\int_{\hat{x}'}^{\hat{x}^0} \hat{T}_{\text{eff}}(x'')dx''} \hat{T}^3_{\text{eff}}(\hat{x}')\,.$$

Boltzmann eq. w/ RTA: solution

$$\left(\hat{p}^{\mu}\hat{\partial}_{\mu}+\hat{\Gamma}^{\rho}{}_{\mu\nu}\hat{p}^{\mu}\hat{p}_{\rho}\frac{\partial}{\partial\hat{p}_{\nu}}\right)f(\hat{x}^{\alpha},\hat{p}_{\beta})=\mathscr{C}[f],$$

$$\mathscr{C}[f]=\frac{\hat{p}_{\mu}\hat{u}^{\mu}(x)}{\tau_r(\hat{x})}\Big(f(\hat{x}^{\alpha},\hat{p}_{\beta})-f_{eq}(\hat{x}^{\alpha},\hat{p}_{\beta})\Big),$$

$$\hat{x}^{\mu}=\{\hat{x}^0,\hat{x}^1\cancel{,x,y}\}$$

$$\hat{g}^{\mu\nu}=\mathrm{diag}\left[e^{2\frac{1-a^2}{1+a^2}\hat{x}^1},-(\hat{x}^0)^{-2}e^{2\frac{1-a^2}{1+a^2}\hat{x}^1},\cancel{-1,-1}\right],$$

"ture" 1+1D system

"false" 1+1D system

$$\hat{T}^2_{\rm eff}(\hat{x}^0)=\frac{(\hat{x}^0)^2}{(\hat{x}^0_i)^2}e^{-\frac{1}{5\bar{\eta}}\int_{\hat{x}^0_i}^{\hat{x}^0}\hat{T}_{\rm eff}(x')dx'}\hat{T}^2_0$$

decay of initial state

$$+\frac{1}{5\bar{\eta}}\int_{\hat{x}^0_i}^{\hat{x}^0}dx'\frac{(\hat{x}^0)^2}{(\hat{x}')^2}e^{-\frac{1}{5\bar{\eta}}\int_{\hat{x}'}^{\hat{x}^0}\hat{T}_{\rm eff}(x'')dx''}\hat{T}^3_{\rm eff}(\hat{x}').$$

approach to equilibrium

Boltzmann eq. w/ RTA: solution

$$\left(\hat{p}^\mu \hat{\partial}_\mu + \hat{\Gamma}^\rho{}_{\mu\nu}\hat{p}^\mu \hat{p}_\rho \frac{\partial}{\partial \hat{p}_\nu}\right) f(\hat{x}^\alpha, \hat{p}_\beta) = \mathscr{C}[f]\,,$$

$$\mathscr{C}[f] = \frac{\hat{p}_\mu \hat{u}^\mu(x)}{\tau_r(\hat{x})}\left(f(\hat{x}^\alpha, \hat{p}_\beta) - f_{eq}(\hat{x}^\alpha, \hat{p}_\beta)\right)\,,$$

$$\hat{x}^\mu = \{\hat{x}^0, \hat{x}^1, x, y\}$$

$$\hat{g}^{\mu\nu} = \mathrm{diag}\left[e^{2\frac{1-a^2}{1+a^2}\hat{x}^1}, -(\hat{x}^0)^{-2} e^{2\frac{1-a^2}{1+a^2}\hat{x}^1}, -1, -1\right],$$

“ture” 1+1D system ✓

“false” 1+1D system

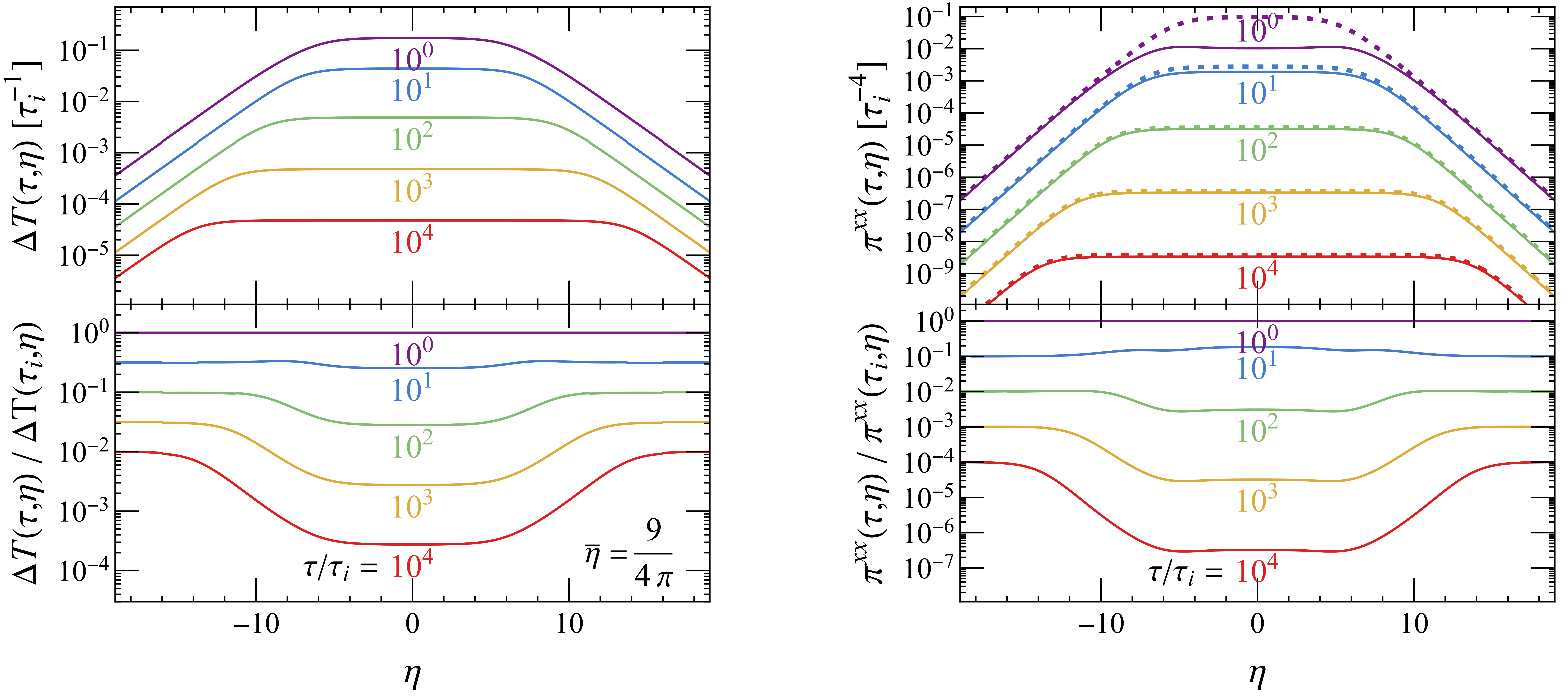
only for *a*=0 (symmetric)

$$f(\hat{x}^0, \hat{p}_1, \hat{p}_T) = D(\hat{x}^0, \hat{x}^0_i) f_0(\hat{x}^0_i, \hat{p}_1, \hat{p}_T) + \frac{1}{5\bar{\eta}} \int_{\hat{x}^0_i}^{\hat{x}^0} d\hat{x}' D(\hat{x}^0, \hat{x}') \hat{T}(\hat{x}') e^{-\frac{\hat{p}_0(\hat{x}')}{\hat{T}(\hat{x}')}}$$

mathematics: simply replacing variables $\{\tau \to \hat{x}^0, \eta \to \hat{x}^1\}$ in the Bjorken solution

physics: non-trivial because *rapidity dependence* is introduced!

Boltzmann eq. w/ RTA: solution

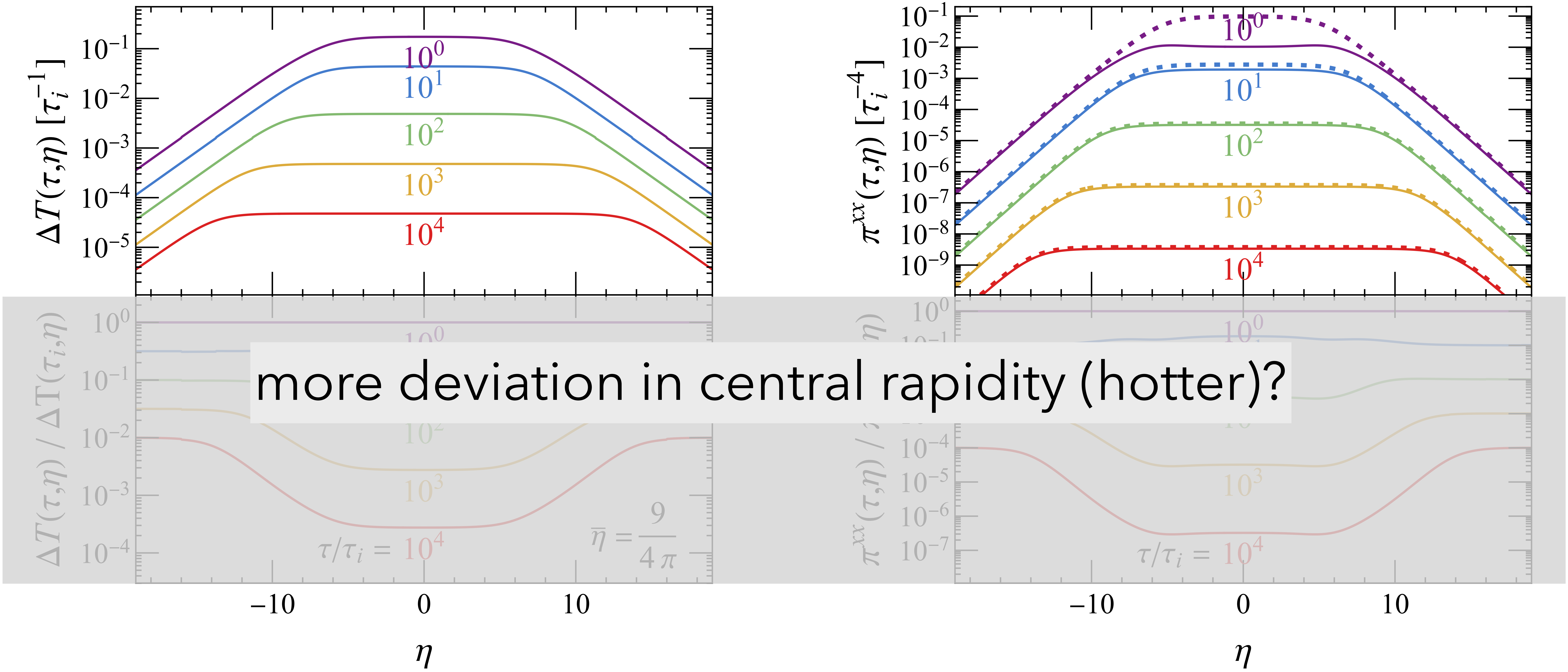


left: [exact solution] - [NS]

solid:[exact solution] dash:[NS]

physics: non-trivial because *rapidity dependence* is introduced!

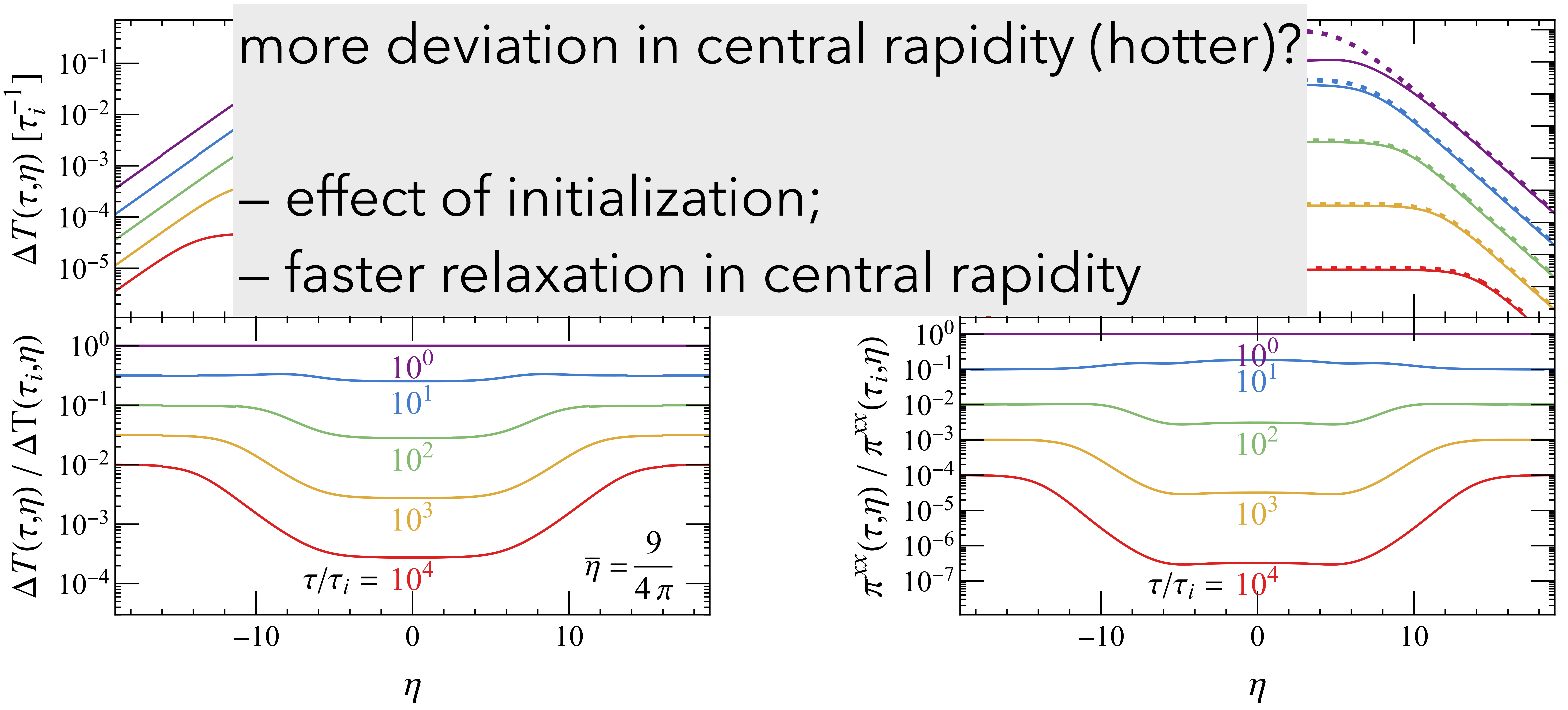
Boltzmann eq. w/ RTA: solution



left: [exact solution] - [NS]   solid:[exact solution] dash:[NS]

physics: non-trivial because *rapidity dependence* is introduced!

Boltzmann eq. w/ RTA: solution



left: [exact solution] - [NS] solid:[exact solution] dash:[NS]

physics: non-trivial because *rapidity dependence* is introduced!

summary and outlook

found analytical solutions in 1+1D expanding systems:

	rapidity dependence	could it be asymmetric?
Hydrodynamics	✓	✓
RTA Boltzmann "true" (1+1)	✓	✓
RTA Boltzmann "false" (1+1)	✓	✗

- anisotropic hydro?
- analytical solutions in magnetohydro?