QCD relations on gravitational form factors and hadron mass decompositions

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KT, JHEP03, 013 ('23)

KT, in preparation

Contents

1. Gravitational form factors

   energy-momentum tensor, GPD

2. $\bar{C}_{q,g}$ at NNLO QCD

   trace anomaly constraints

3. Mass decompositions at NNLO QCD

   N, π

Symmetric energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi \;+\; F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2$$

$$T^{\mu\nu} = T^{\nu\mu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

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$$\left(= \frac{1}{2}\bar{\psi}\gamma^{\mu} i\partial^{\nu}\psi - F^{\mu\rho}\partial^{\nu}A_{\rho} + \frac{\eta^{\mu\nu}}{4}F^2 + \partial_{\lambda}X^{[\lambda\mu]\nu} \right)$$

$$\underbrace{}$$

$$\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial^{\nu}\phi_n - \eta^{\mu\nu}\mathcal{L}$$

$$T^{\mu\nu} = T^{\nu\mu}$$

$$\partial_{\mu}T^{\mu\nu} = 0$$

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi \;+\; F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^{2}$$

trasnlation	$T^{\mu\nu}$	$\partial_\mu T^{\mu\nu} = 0$
Lorentz tr.	$M^{\mu\nu\lambda} = x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu}$	$\partial_\mu M^{\mu\nu\lambda} = 0$
scale tr.	$D^\mu = x_\nu T^{\mu\nu}$	$\partial_\mu D^\mu = T^\mu_\mu$
conformal tr.	$C^{\mu\nu} = \left(2x^\rho x^\nu - \eta^{\rho\nu}x^2\right)T_\rho{}^\mu$	$\partial_\mu C^{\mu\nu} = 2x^\nu T^\mu_\mu$

Symmetric energy-momentum tensor

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$$\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)}\partial^\nu \phi_n - \eta^{\mu\nu}\mathcal{L}$$

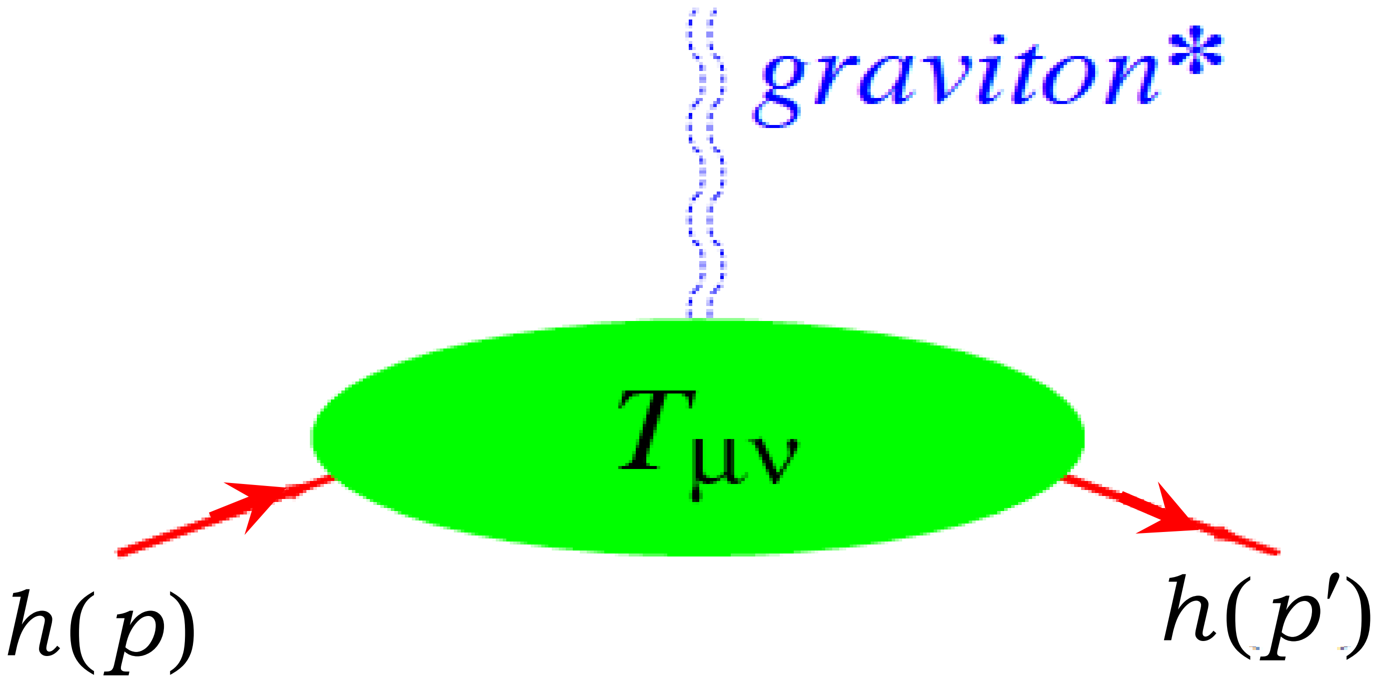
$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}(x)}\bigg|_{g^{\mu\nu}\to\eta^{\mu\nu}}$$

$$T^{\mu\nu} = T^{\nu\mu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi \ + \ F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2$$

$$\equiv \quad T_q^{\mu\nu} \quad + \quad T_g^{\mu\nu}$$



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$$\langle N(p')|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)M\eta^{\mu\nu}\Big]u(p)$$

$$P = \frac{p+p'}{2}$$

$$\Delta = p' - p$$

$$t = \Delta^2$$

Ji, PRL78, 610 ('97)

Polyakov, Schweitzer, IJMPA33, 1830025('18)

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$$A_q(0) + A_g(0) = 1$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle = 2p^\mu p^\nu$$

$$\frac{1}{2}\Big(A_q(0) + B_q(0) + A_g(0) + B_g(0)\Big) = \frac{1}{2}$$

$$B_q(0) + B_g(0) = 0$$

$$\frac{\langle N(p)\, S|J^i|N(p)\, S\rangle}{\langle N(p)\, S|N(p)\, S\rangle} = \frac{1}{2}S^i$$

$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{+jk}$$

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

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$$\bar{C}_q(t) + \bar{C}_g(t) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$D \equiv D_q(0) + D_g(0)$$

"D term": the last unknown global property

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2} \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

mass & energy distribution

angular momentum distribution

$$\langle N(p')|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)M\eta^{\mu\nu}\Big]u(p)$$

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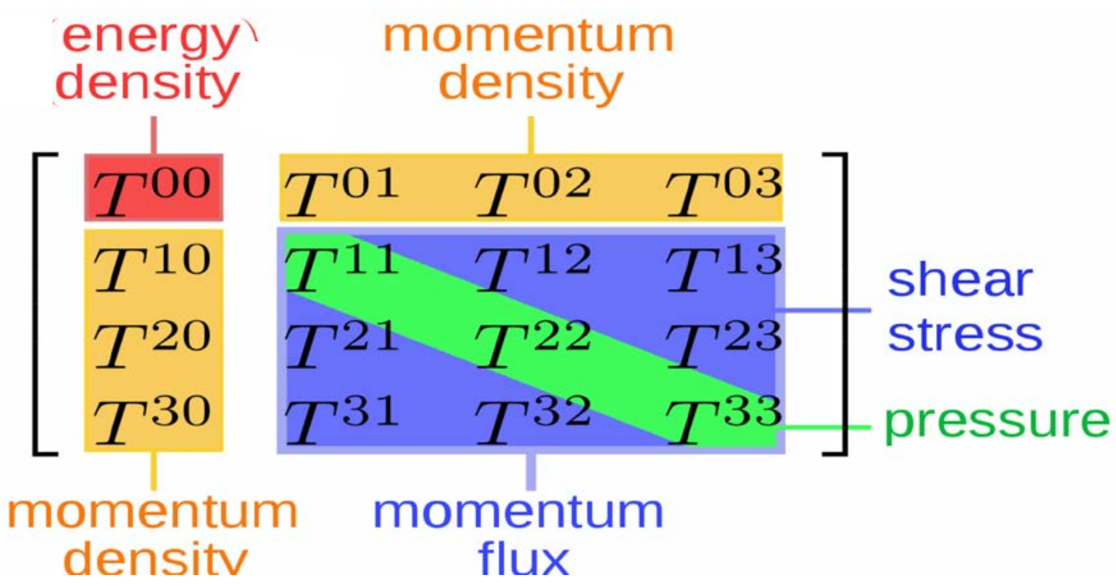
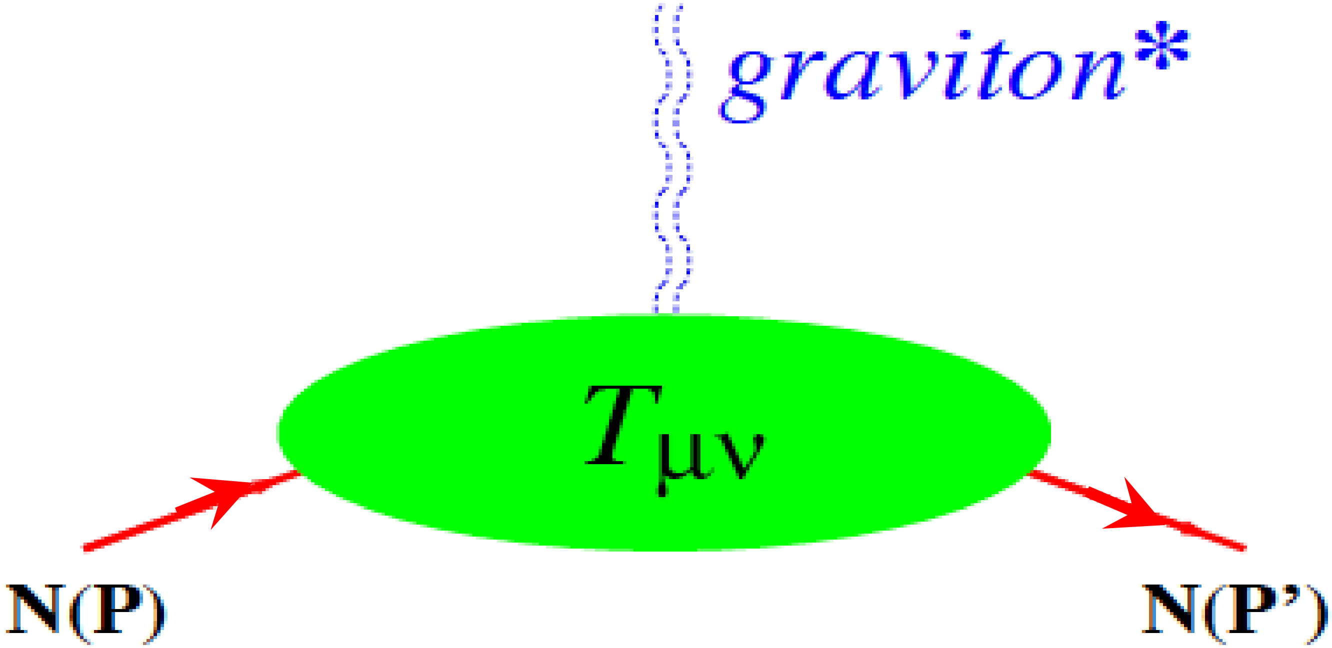
force & pressure distribution

mass & pressure distribution

$$T^{\mu\nu} =$$





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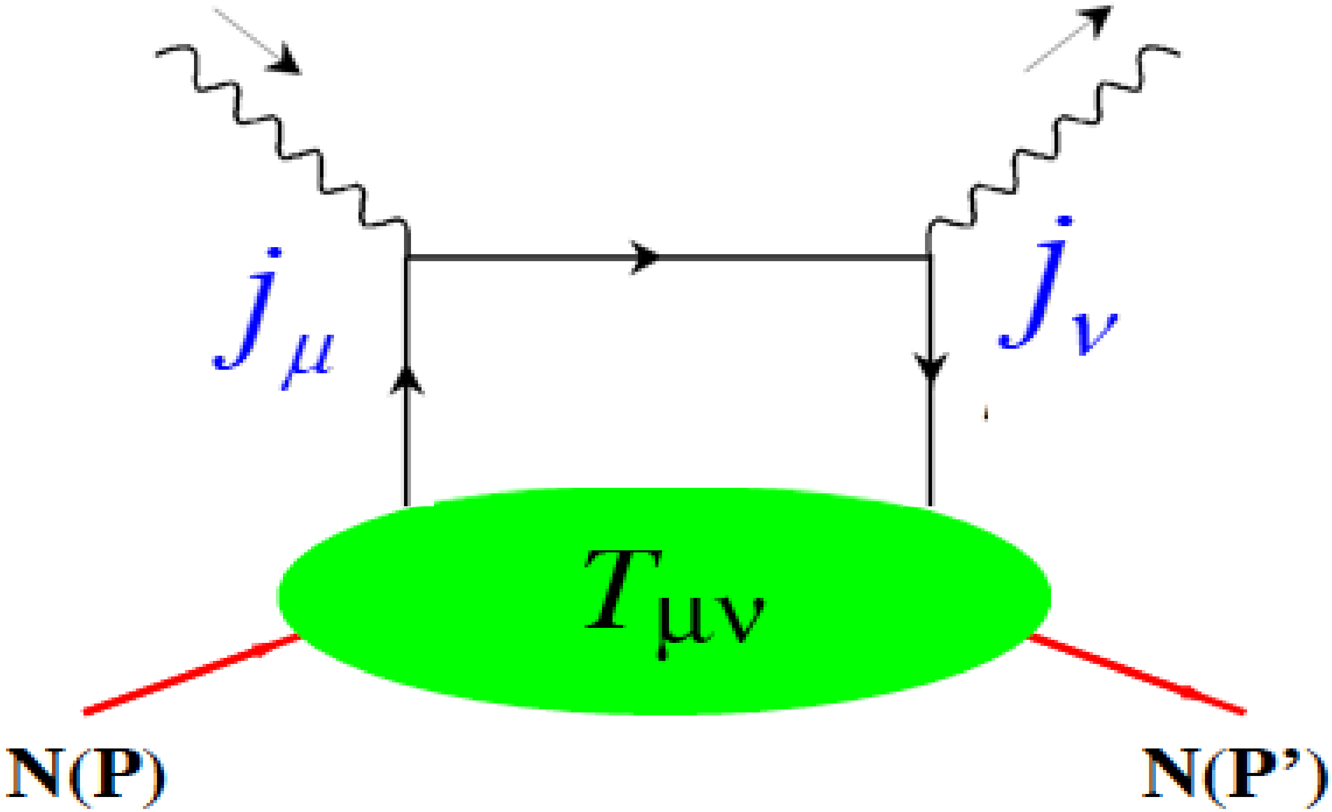

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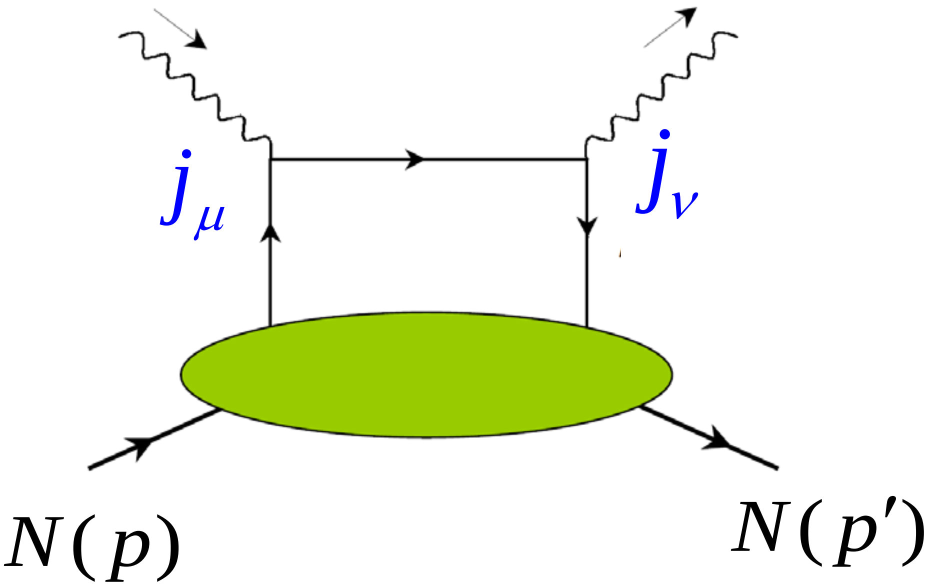
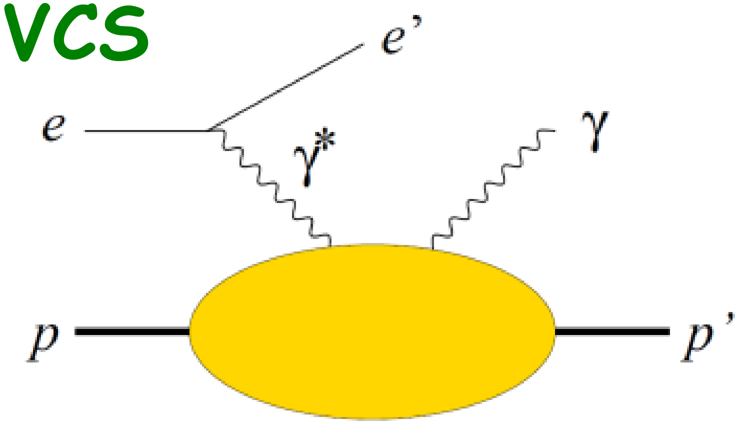


$$j_\mu(x)\, j_\nu(0) \sim \sum_i C_i(x) O_i(0)$$

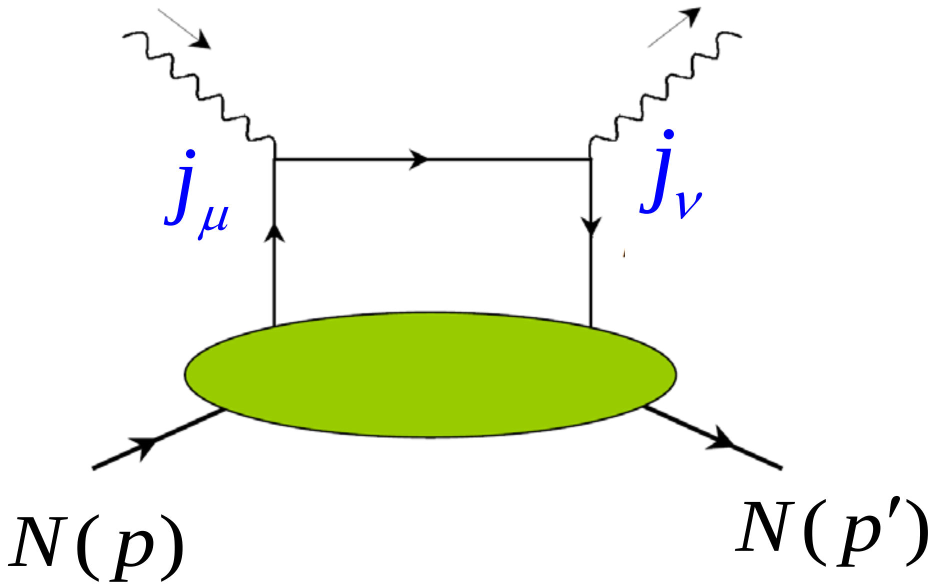
$$C^q_{\mu\nu;\alpha\beta}(x) T_q^{\alpha\beta}$$

$$C^g_{\mu\nu;\alpha\beta}(x) T_g^{\alpha\beta}$$

DVCS



JLab, EIC…



$$j_\mu(x)\, j_\nu(0) \sim \sum_i C_i(x) O_i(0)$$

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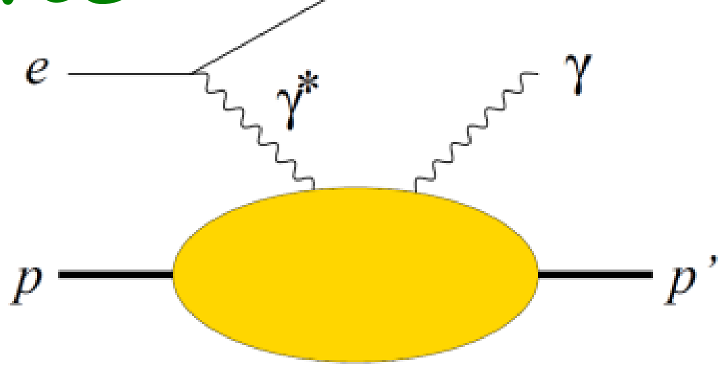
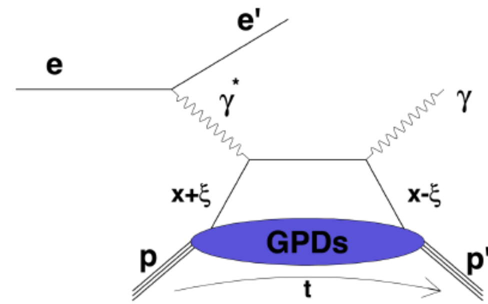
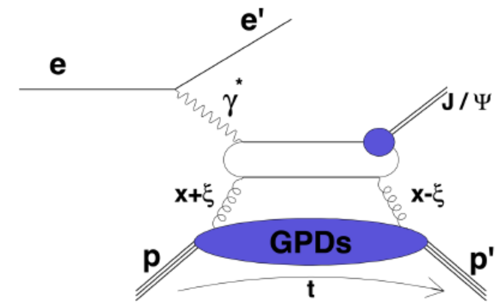
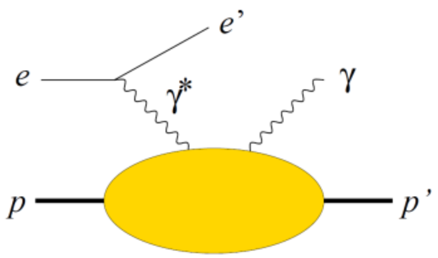
DVCS







JLab, EIC…



DVCS

$$P = \frac{p+p'}{2}$$

JLab, HERMES, COMPASS, EIC

$$\int \frac{dz^-}{2\pi} e^{ixPz} \langle N(p') | \bar{\psi}(-\frac{z^-}{2}) \gamma^+ \psi(\frac{z^-}{2}) | N(p) \rangle = \frac{1}{P^+}\left[H^q(x,\eta,t)\bar{u}(p')\gamma^+ u(p) + E^q(x,\eta,t)\bar{u}(p')\frac{i\sigma^{+\alpha}(p'-p)_\alpha}{2M} u(p)\right]$$

$$\int \frac{dz^-}{2\pi} e^{ixPz} \langle N(p') | \bar{\psi}(-\frac{z^-}{2}) \gamma^+ \gamma_5 \psi(\frac{z^-}{2}) | N(p) \rangle = \frac{1}{P^+}\left[\tilde{H}^q(x,\eta,t)\bar{u}(p')\gamma^+ \gamma_5 u(p) + \tilde{E}^q(x,\eta,t)\bar{u}(p')\frac{\gamma_5 (p'-p)^+}{2M} u(p)\right]$$

$$\Delta^\mu = p'^\mu - p^\mu \to 0$$

$$\left(t = \Delta^2 \to 0, \quad \eta = \frac{-\Delta \cdot n}{2P \cdot n} \to 0 \right)$$

$$H^q(x,0,0) = q(x)$$

$$\int_{-1}^{1} dx H^q(x,\eta,t) = F_1^q(t), \qquad \int_{-1}^{1} dx E^q(x,\eta,t) = F_2^q(t)$$

$$\int_{-1}^{1} dx\, x H^q(x,\eta,t) = A_q(t) + 4\eta^2 D_q(t), \qquad \int_{-1}^{1} dx\, x E^q(x,\eta,t) = B_q(t) - 4\eta^2 D_q(t)$$

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

mass & energy distribution

angular momentum distribution

$$\langle N(p')\,|\,T_{q,g}^{\mu\nu}\,|\,N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)M\eta^{\mu\nu}\Big]u(p)$$

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force & pressure distribution

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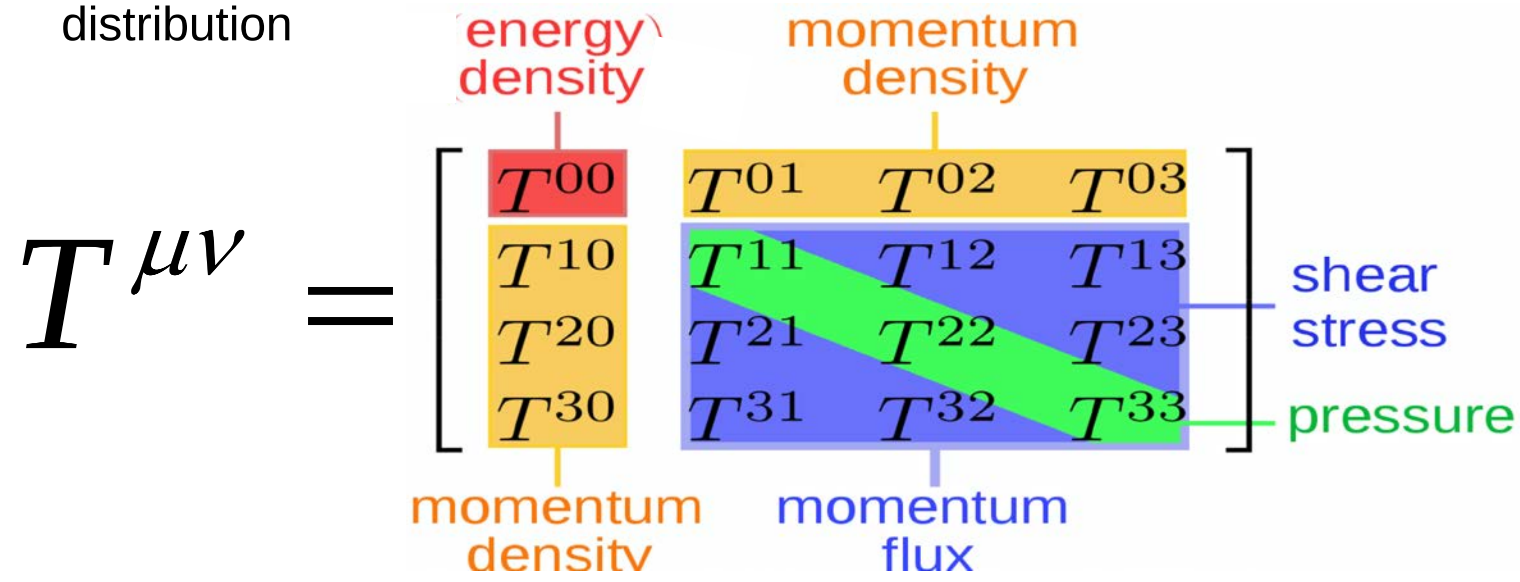
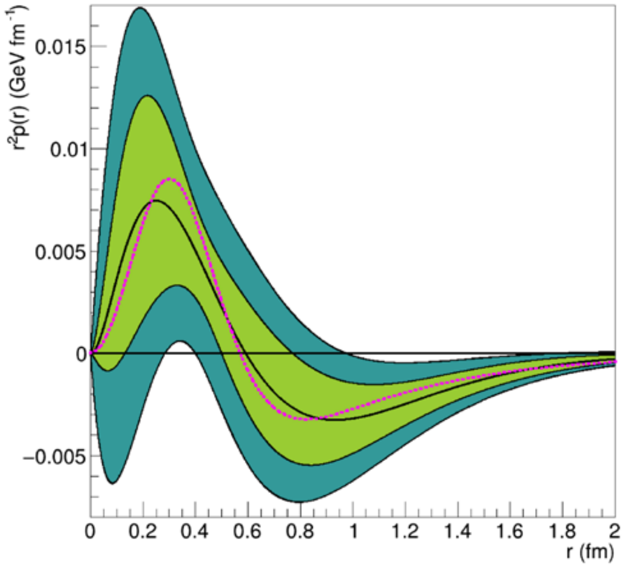
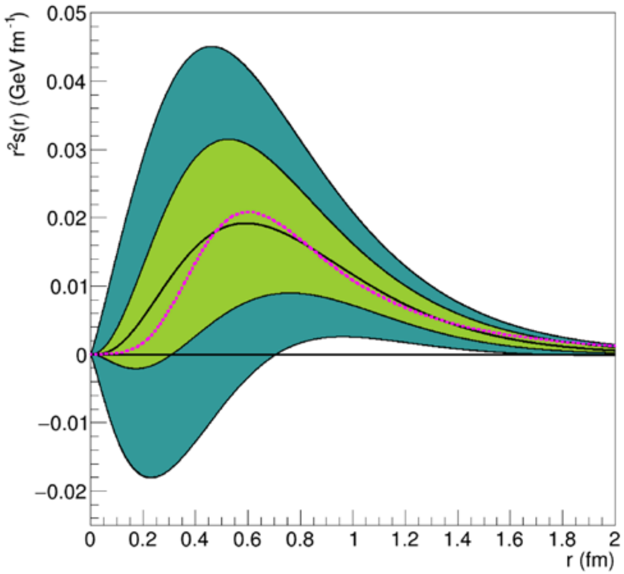
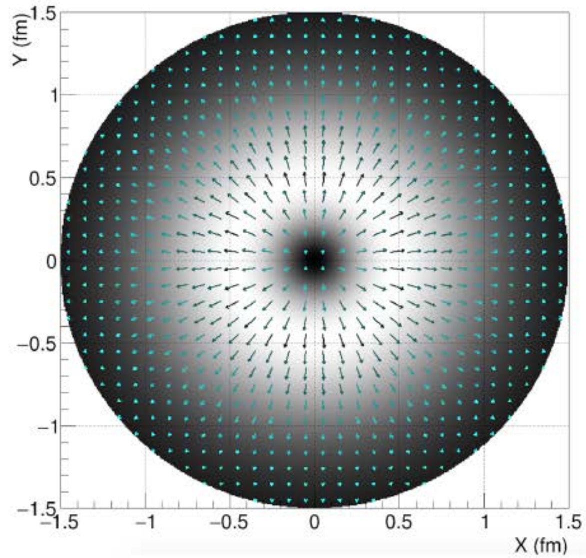


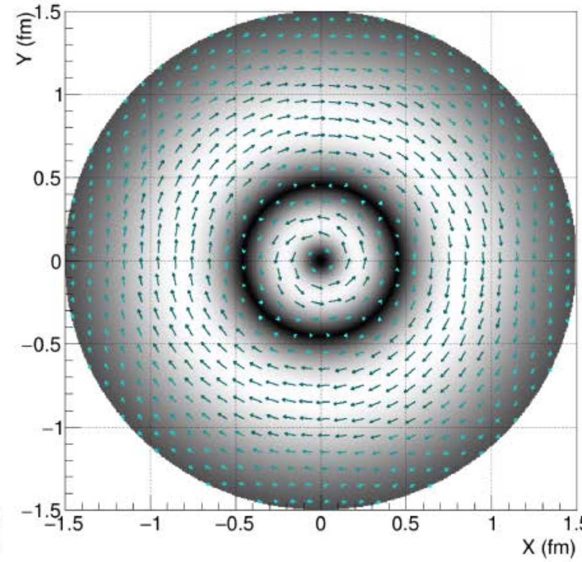
V. D. Burkert et al, Nature 557 ('18) 396

V. D. Burkert et al, 2303.08347









$$\langle N(p')|T^{ik}|N(p)\rangle \sim \left(\Delta^i\Delta^k - \delta^{ik}\mathbf{\Delta}^2\right)D(t), \qquad \left\langle T^{ij}\right\rangle(r) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right)s(r) + \delta^{ij}\,p(r)$$

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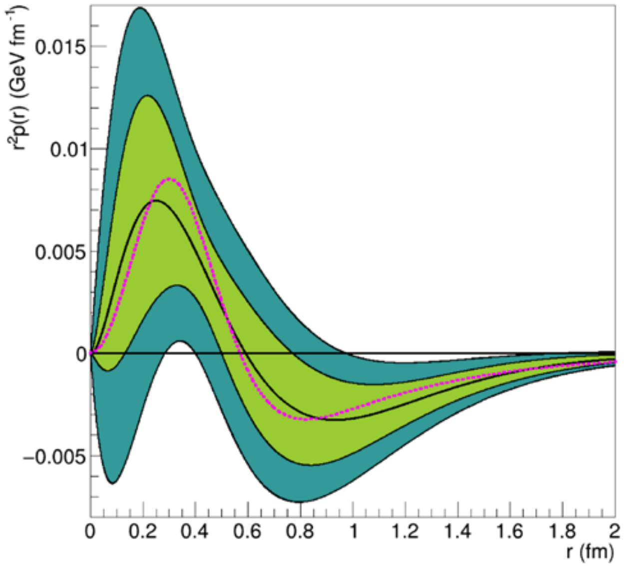

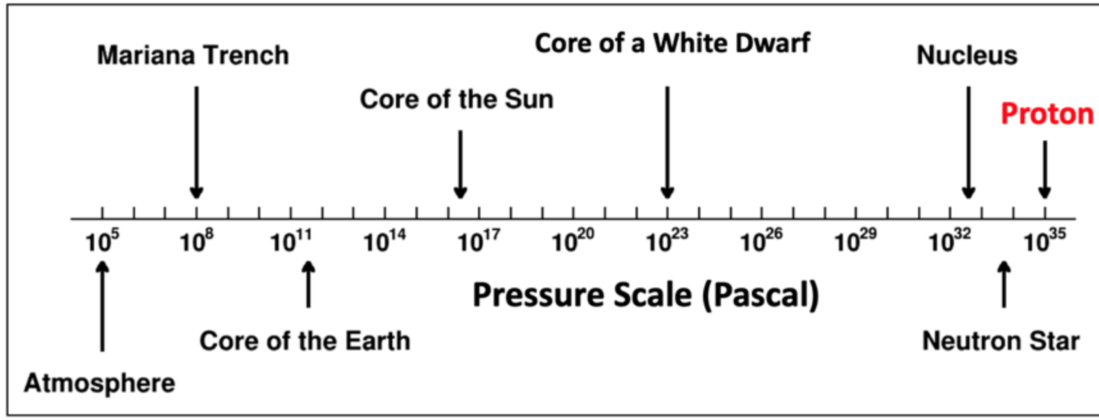

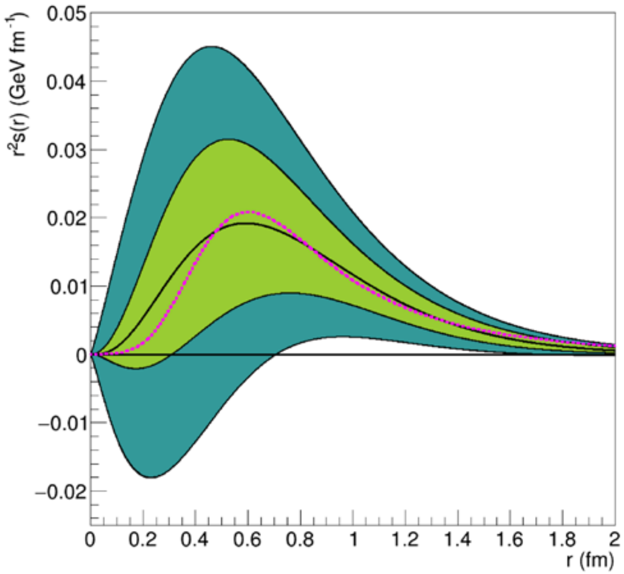

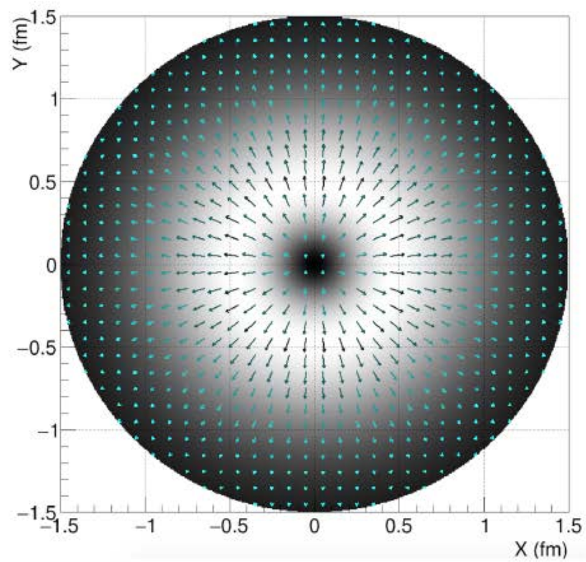

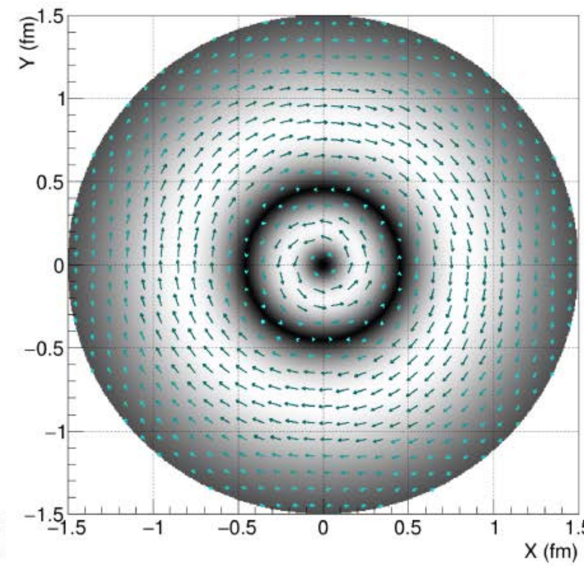

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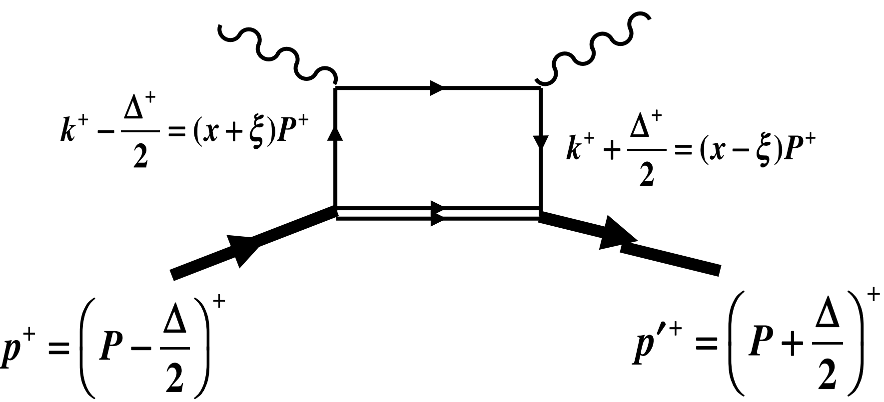
Kumano, Song, Teryaev, PRD97, 014020 ('18) ,1711.08088

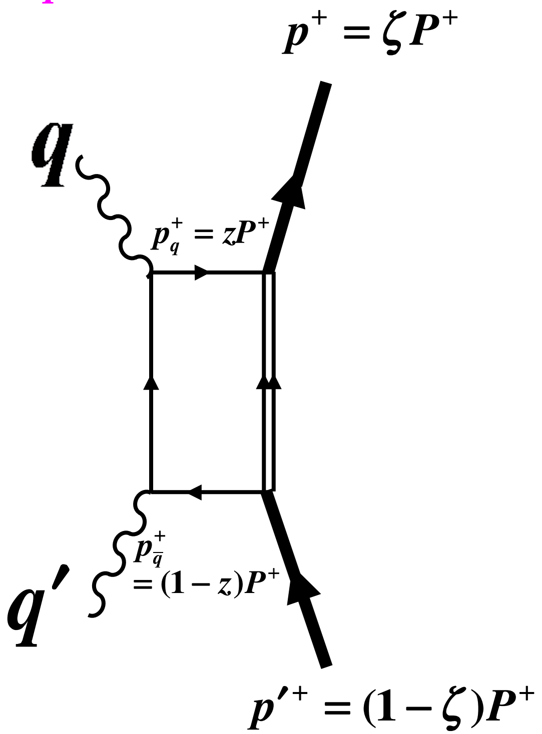
GPD: $$H_q(x,\xi,t) = \int \frac{dy^-}{4\pi} e^{ixP^+y^-} \langle \pi^0(p') | \bar{\psi}(-y/2)\gamma^+\psi(y/2) | \pi^0(p) \rangle \Big|_{y^+=0,\vec{y}_\perp=0}, \qquad P^+ = \frac{(p+p')^+}{2}$$

GDA: $$\Phi_q(z,\zeta,s) = \int \frac{dy^-}{2\pi} e^{izP^+y^-} \langle \pi^0(p)\pi^0(p') | \bar{\psi}(-y/2)\gamma^+\psi(y/2) | 0 \rangle \Big|_{y^+=0,\vec{y}_\perp=0}$$

$H_q^h(x,\xi,t)$ ⟷ $\Phi_q^{hh}(z,\zeta,W^2)$

s-t crossing





$\gamma\gamma^* \to \pi^0\pi^0$

Kumano, Song, Teryaev, PRD97, 014020 ('18) ,1711.08088

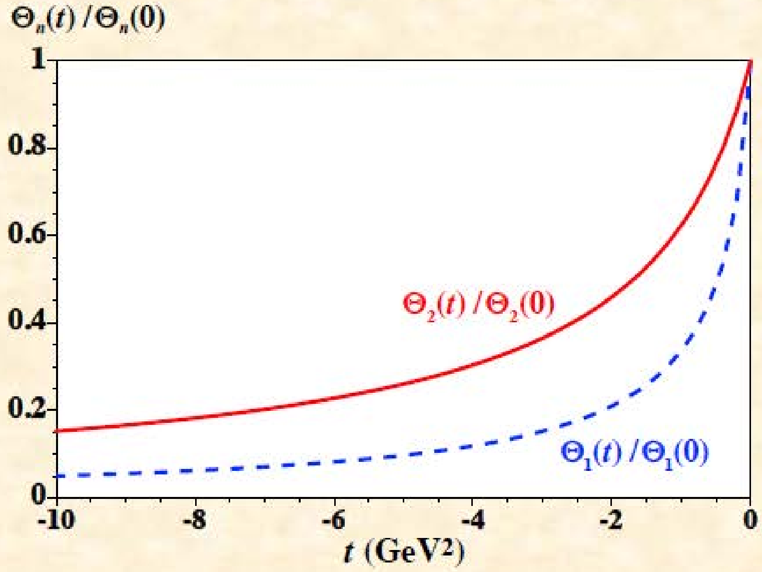
Spacelike gravitational form factors and radii for pion

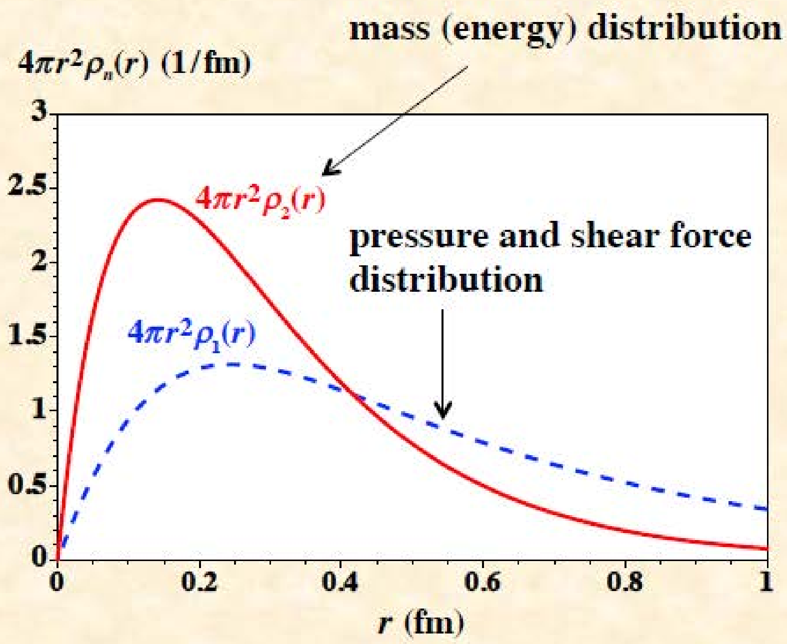
$$F(s)=\Theta_1(s),\ \Theta_1(s),\quad F(t)=\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im}\,F(s)}{\pi(s-t-i\varepsilon)},\quad \rho(r)=\frac{1}{(2\pi)^3}\int d^3q e^{-i\vec{q}\cdot\vec{r}} F(q)=\frac{1}{4\pi^2}\frac{1}{r}\int_{4m_\pi^2}^{\infty} ds\, e^{-\sqrt{s}r}\,\mathrm{Im}\,F(s)$$

This is the first report on gravitational radii of hadrons from actual experimental measurements.

$\sqrt{\langle r^2 \rangle_{\text{mass}}} = 0.32 \sim 0.39$ fm, $\sqrt{\langle r^2 \rangle_{\text{mech}}} = 0.82 \sim 0.88$ fm ← First finding on gravitational radius from actual experimental measurements

$\Leftrightarrow \sqrt{\langle r^2 \rangle_{\text{charge}}} = 0.672 \pm 0.008$ fm





$$\Theta_2(t)=4A^\pi(t),\quad \Theta_1(t)=-D^\pi(t)$$

Kumano, Song, Teryaev, PRD97, 014020 ('18) ,1711.08088

Spacelike gravitational form factors and radii for pion

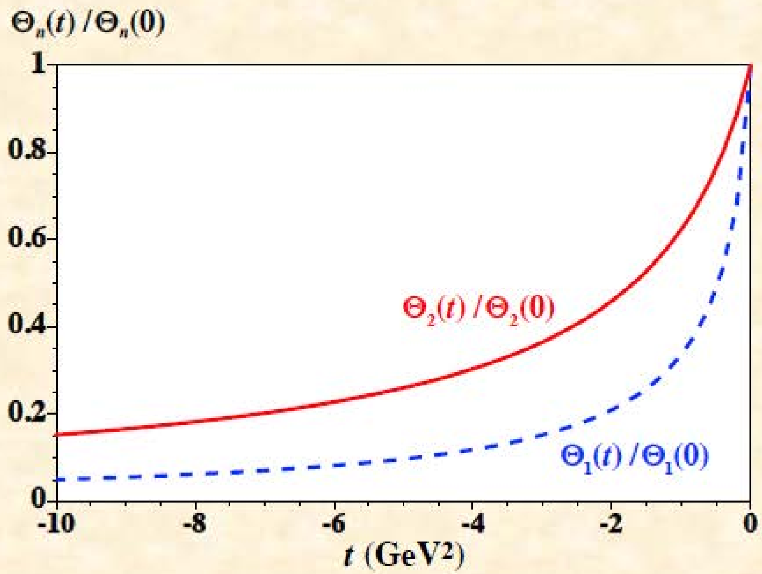
$$F(s)=\Theta_1(s),\ \Theta_1(s),\quad F(t)=\int_{4m_\pi^2}^{\infty} ds\,\frac{\mathrm{Im}\,F(s)}{\pi(s-t-i\varepsilon)},\quad \rho(r)=\frac{1}{(2\pi)^3}\int d^3q\,e^{-i\vec{q}\cdot\vec{r}}F(q)=\frac{1}{4\pi^2}\frac{1}{r}\int_{4m_\pi^2}^{\infty} ds\,e^{-\sqrt{s}r}\,\mathrm{Im}\,F(s)$$

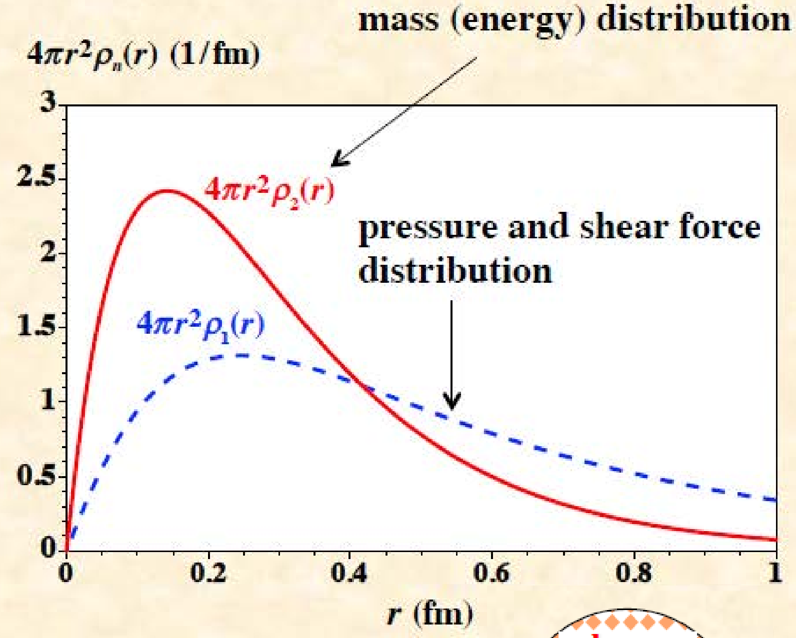
This is the first report on gravitational radii of hadrons from actual experimental measurements.

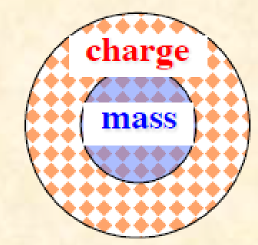
$$\sqrt{\langle r^2\rangle_{\rm mass}}=0.32\sim0.39\ \text{fm},\quad \sqrt{\langle r^2\rangle_{\rm mech}}=0.82\sim0.88\ \text{fm}$$

← First finding on gravitational radius from actual experimental measurements

$$\Leftrightarrow\ \sqrt{\langle r^2\rangle_{\rm charge}}=0.672\pm0.008\ \text{fm}$$



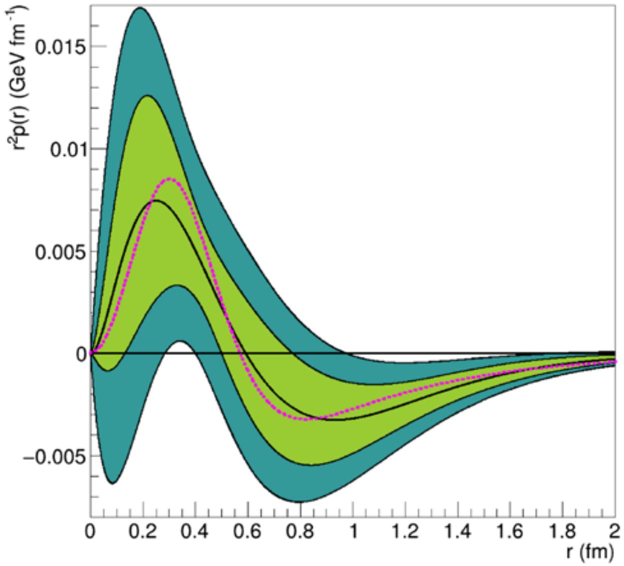


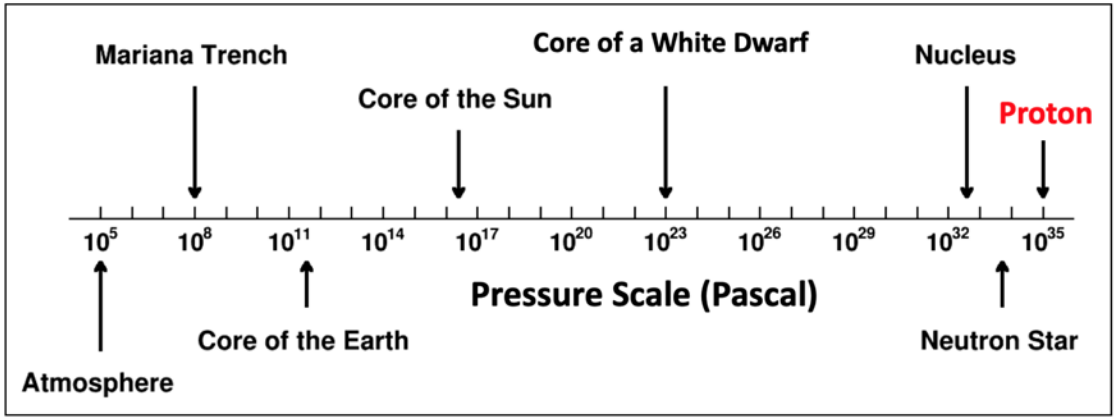


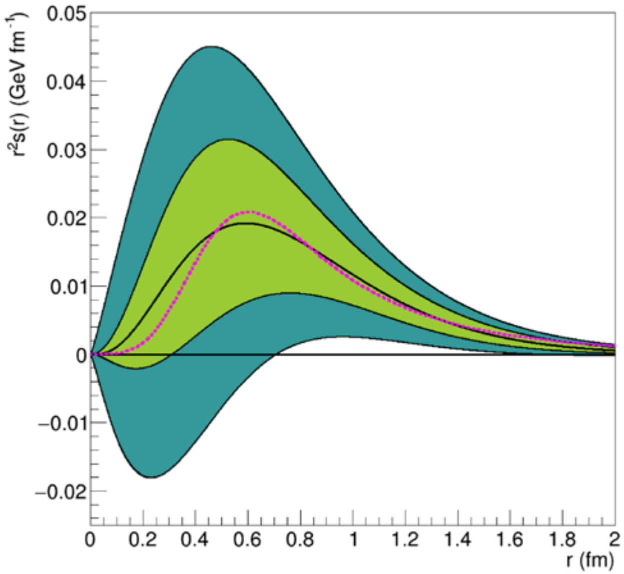
$$\Theta_2(t)=4A^\pi(t),\qquad \Theta_1(t)=-D^\pi(t)$$

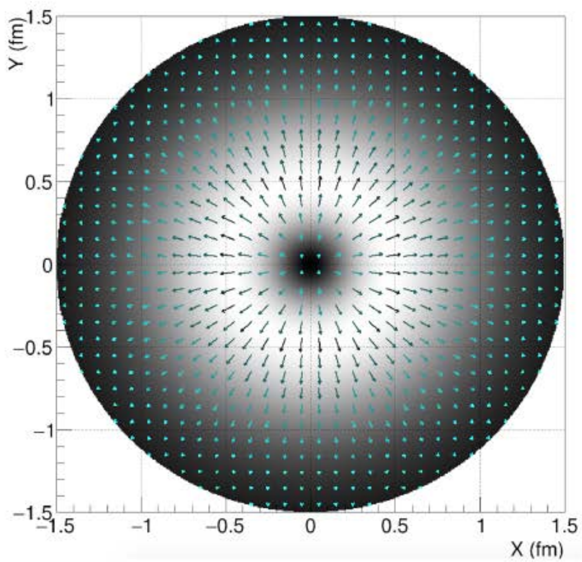
V. D. Burkert et al, Nature 557 ('18) 396

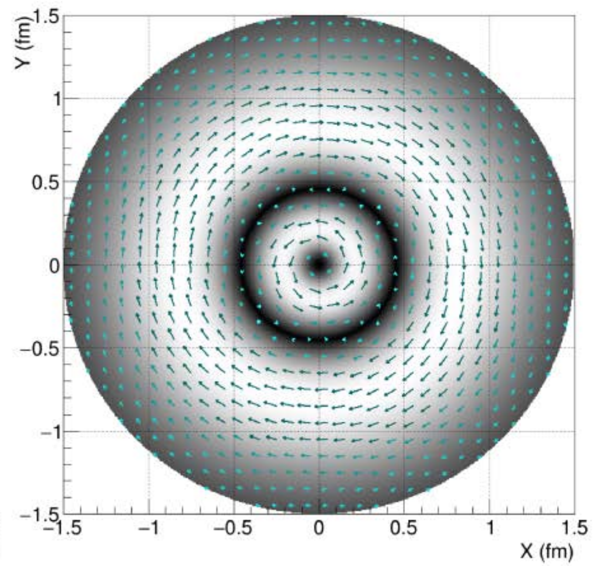
V. D. Burkert et al, 2303.08347







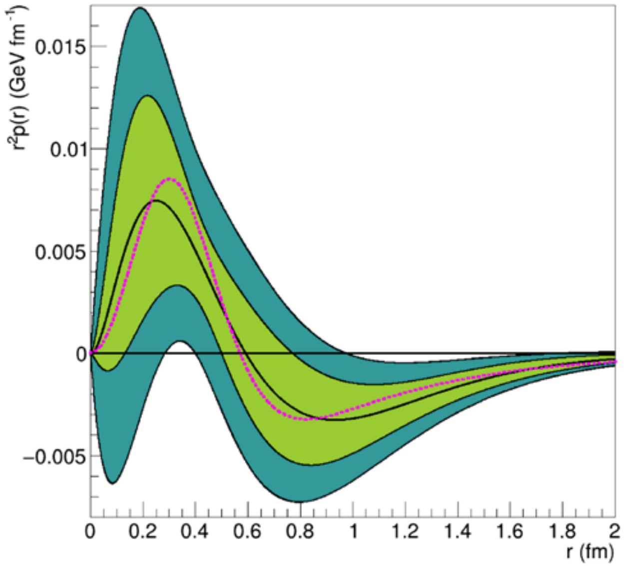


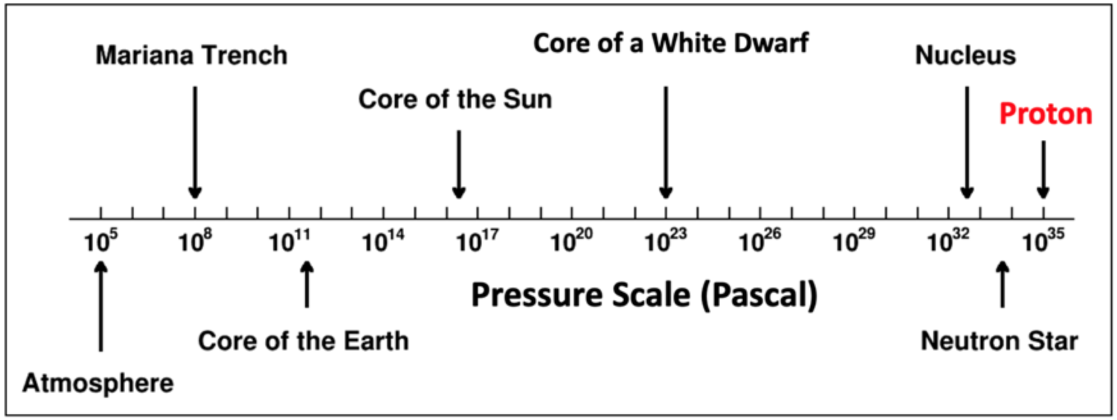


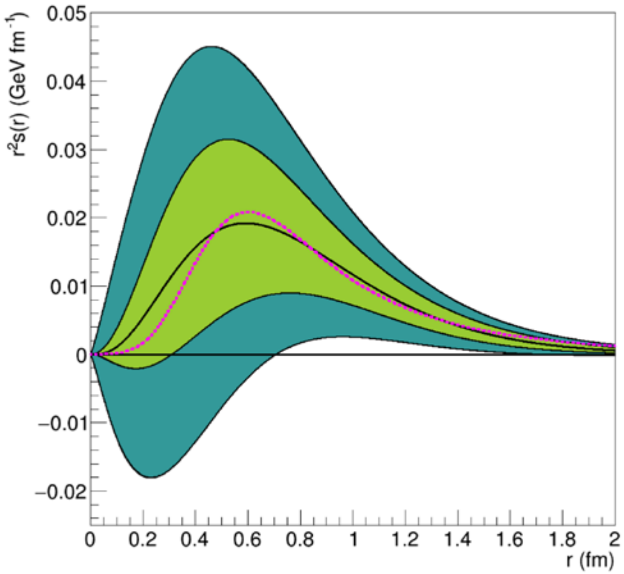
$$\langle N(p') | T^{ik} | N(p) \rangle \sim \left(\Delta^i \Delta^k - \delta^{ik} \mathbf{\Delta}^2 \right) D(t), \qquad T^{ij}(r) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3} \delta^{ij} \right) s(r) + \delta^{ij} \, p(r)$$

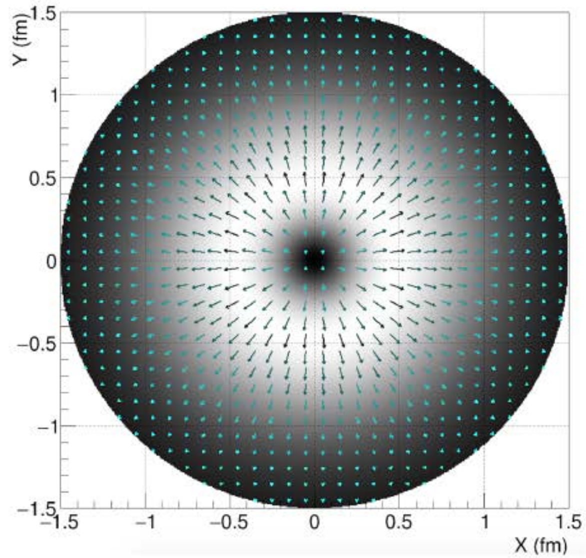
V. D. Burkert et al, Nature 557 ('18) 396

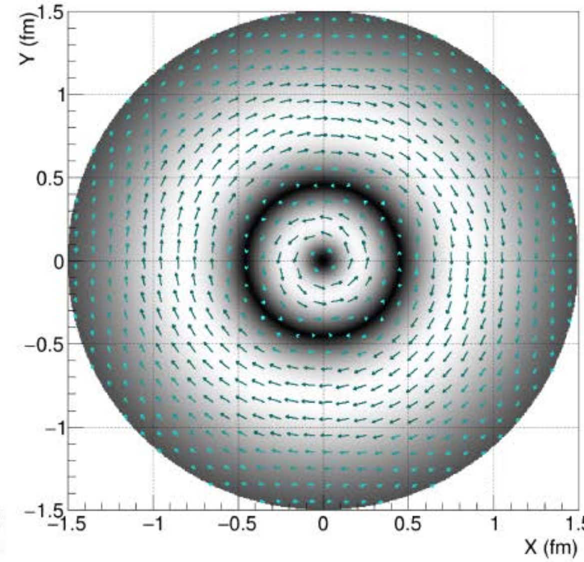
V. D. Burkert et al, 2303.08347











$$\langle N(p')|T_q^{ik}|N(p)\rangle \sim \left(\Delta^i\Delta^k - \delta^{ik}\mathbf{\Delta}^2\right)D_q(t) - 4M^2\delta^{ik}\bar{C}_q(t)$$

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$\langle N(p')|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)M\eta^{\mu\nu}\Big]u(p)$$

$$P = \frac{p+p'}{2}$$

$$\Delta = p' - p$$

$$t = \Delta^2$$

$$A_q(0) + A_g(0) = 1$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle = 2p^\mu p^\nu$$

$$\frac{1}{2}\Big(A_q(0) + B_q(0) + A_g(0) + B_g(0)\Big) = \frac{1}{2}$$

$$B_q(0) + B_g(0) = 0$$

$$\frac{\langle N(p)\,S|J^i|N(p)\,S\rangle}{\langle N(p)\,S|N(p)\,S\rangle} = \frac{1}{2}S^i$$

$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{+jk}$$

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

$$\bar{C}_q(t) + \bar{C}_g(t) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$D \equiv D_q(0) + D_g(0)$$

"D term": the last unknown global property

① mass decomposition

Ji, PRD52 271 ('95)

Lorce, Moutarde, Trawinski, EPJC79, 89 ('19)

Metz, Pasquini, Rodini, PRD102, 114042 ('20)

Ji, Liu, Schafer, NPB971, 115537 ('21)

② pressure

$$-\bar{C}_{q,g}\frac{M}{V}$$

Lorce, EPJC78, 120 ('18)

Liu, PRD104, 076010 ('21)

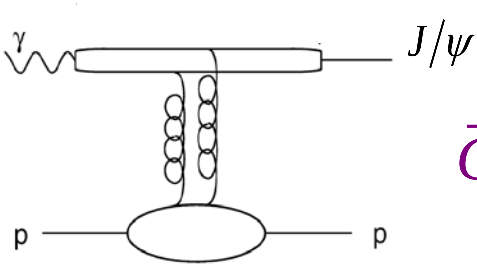
③ nucleon's transverse spin sum rule

$$J_{q,g}=\frac{1}{2}(A_{q,g}+B_{q,g})+\frac{p^3}{2(p^0+M)}\bar{C}_{q,g}$$

Hatta, KT, Yoshida,

JHEP 02 ('13) 003

④ $\gamma p \to J/\psi\ p$ near threshold JLab, EIC



$$\bar{C}_g\ \left(=-\bar{C}_q\right)$$

Y. Hatta, D. Yang, PRD98, 074003

Y. Hatta, A. Rajan, D. Yang,

PRD100, 014032

Studies for $\bar{C}_{q,g}$ themselves

QCD EOMs $(i\not{D} - m)\psi = 0, \quad D_\nu F^{\mu\nu} = g\bar{\psi}\gamma^\mu\psi$

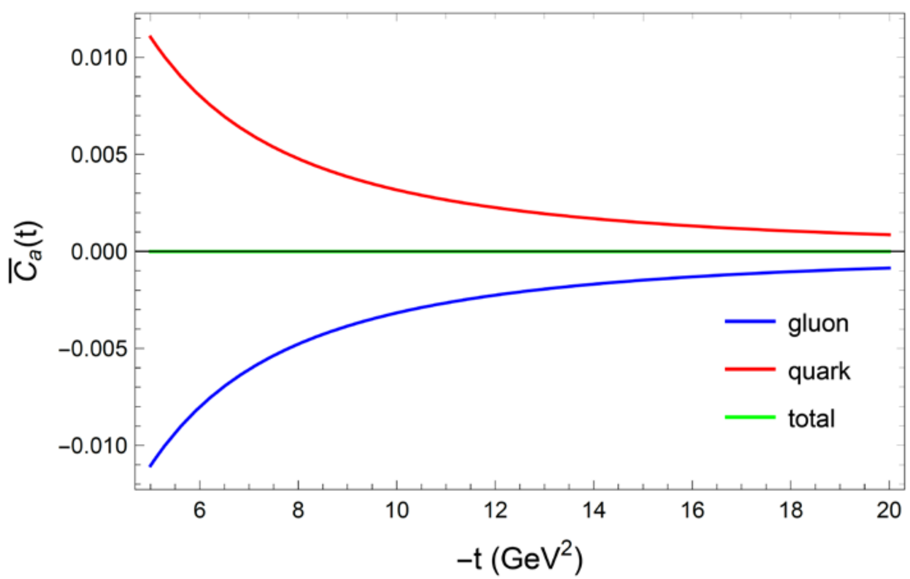
$$\partial_\nu T_q^{\mu\nu} = -\bar{\psi} g F^{\mu\nu}\gamma_\nu\psi, \quad \partial_\nu T_g^{\mu\nu} = -F_a^{\mu\nu} D_{ab}^\rho F_{\rho\nu}^b$$

KT, PRD98, 034009 ('18)

$$\Delta^\mu \bar{u}(p')u(p)M\bar{C}_q(t) = \langle N(p')|\bar{\psi} i g F^{\mu\nu}\gamma_\nu\psi|N(p)\rangle$$

$$\Delta^\mu \bar{u}(p',S')u(p,S)M\bar{C}_g(t) = \langle N(p')|F_a^{\mu\nu} i D_{ab}^\rho F_{\rho\nu}^b|N(p)\rangle$$

pQCD for large t



Tong, Ma, Yuan, PLB823, 136751 ('21)

Tong, Ma, Yuan, JHEP10, 046 ('22)

① mass decomposition

Ji, PRD52 271 ('95)

Lorce, Moutarde, Trawinski, EPJC79, 89 ('19)

Metz, Pasquini, Rodini, PRD102, 114042 ('20)

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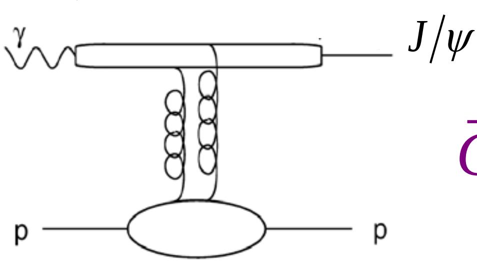
Hatta, KT, Yoshida, JHEP 02 ('13) 003

④ $\gamma p \to J/\psi\ p$ near threshold JLab, EIC



$$\bar{C}_g \ \left(=-\bar{C}_q\right)$$

Y. Hatta, D. Yang, PRD98, 074003

Y. Hatta, A. Rajan, D. Yang, PRD100, 014032

at $t = 0$:

$\bar{C}_q(0, \mu \sim 0.4 \text{ GeV}) = 0.25$ **Bag model** [Ji, Melnitchouk, Song, PRD56, 5511 ('97)]

$\bar{C}_q(0, \mu = 2 \text{ GeV}) \approx -0.11$ **Phenomenological** [Lorce, EPJC78, 120 ('18)]

$\bar{C}_q(0, \mu \sim 0.63 \text{ GeV}) = 0.014$ **Instanton** [Polyakov, Son, JHEP09, 156 ('18)]

$\bar{C}_q(0, \mu = 1 \text{ GeV}) = -0.021 \pm 0.008$ **LCSR** [Azizi,Ozdem, EPJC80, 104 ('20)]

$\bar{C}_q(0, \mu \to \infty) \simeq -0.15$ **Trace anomaly** [Hatta, Rajan, KT, JHEP12, 008 ('18)]

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$\langle N(p')|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)M\eta^{\mu\nu}\Big]u(p)$$

$$P = \frac{p+p'}{2}$$

$$\Delta = p' - p$$

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$$A_q(0) + A_g(0) = 1$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle = 2p^\mu p^\nu$$

$$\frac{1}{2}\Big(A_q(0) + B_q(0) + A_g(0) + B_g(0)\Big) = \frac{1}{2}$$

$$\frac{\langle N(p)\, S|J^i|N(p)\, S\rangle}{\langle N(p)\, S|N(p)\, S\rangle} = \frac{1}{2}S^i$$

$$B_q(0) + B_g(0) = 0$$

$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{+jk}$$

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

$$\bar{C}_q(t) + \bar{C}_g(t) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$D \equiv D_q(0) + D_g(0)$$

"D term": the last unknown global property

$$\langle N(p) | T^{\mu\nu}_{q,g} | N(p) \rangle = \bar{u}(p) \left[A_{q,g}(0) \gamma^{(\mu} p^{\nu)} + \bar{C}_{q,g}(0) M \eta^{\mu\nu} \right] u(p)$$

$$\eta_{\mu\nu} \langle N(p) | T^{\mu\nu}_{q,g} | N(p) \rangle = 2M^2 \left(A_{q,g}(0) + 4\bar{C}_{q,g}(0) \right)$$

$$\bar{C}_q(0) \ \left(= -\bar{C}_g(0) \right) = -\frac{1}{4} A_q(0) + \frac{1}{8M^2} \langle N(p) | \eta_{\mu\nu} T^{\mu\nu}_q | N(p) \rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2} \bar{\psi} \gamma^{(\mu} i \overleftrightarrow{D}^{\nu)} \psi} + \boxed{F^{\mu\rho} F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4} F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$\langle N(p)|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

$$\eta_{\mu\nu}\langle N(p)|T_{q,g}^{\mu\nu}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T_q^{\mu\nu}|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2} \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$\left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

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$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T^{\mu\nu}_q|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_\rho{}^\nu + \frac{\eta^{\mu\nu}}{4}F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad \left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$$\boxed{\eta_{\mu\nu}T^{\mu\nu}_{q,g}}$$

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

$$\eta_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

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$\eta_{\mu\nu}T^{\mu\nu}_{q,g}$ **1&2-loop** Hatta, Rajan, KT, JHEP 12 ('18) 008

$$\langle N(p)|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

$$\eta_{\mu\nu}\langle N(p)|T_{q,g}^{\mu\nu}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T_q^{\mu\nu}|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2} \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad \left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$\eta_{\mu\nu}T_{q,g}^{\mu\nu}$	**1&2-loop**	Hatta, Rajan, KT, JHEP 12 ('18) 008
	3-loop (& all orders)	KT, JHEP 01 ('19) 120

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

$$\eta_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T^{\mu\nu}_q|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad \left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$\eta_{\mu\nu}T^{\mu\nu}_{q,g}$

1&2-loop Hatta, Rajan, KT, JHEP 12 ('18) 008

3-loop (& all orders) KT, JHEP 01 ('19) 120

4-loop Ahmed, Chen, Czakon, JHEP 01 ('23) 077

trace anomaly separately for q, g

Hatta, Rajan, KT, JHEP 12 ('18) 008

$$\eta_{\mu\nu} T_q^{\mu\nu} = m\bar{\psi}\psi + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi + \frac{1}{3}n_f F^2\right)$$

$$\eta_{\mu\nu} T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{\psi}\psi - \frac{11}{6}C_A F^2\right)$$

$$\eta_{\mu\nu} T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$C_A = N_c\ , \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$

trace anomaly separately for q, g $\overline{\text{MS}}$ scheme Hatta, Rajan, KT, JHEP

$$\eta_{\mu\nu}T_q^{\mu\nu} = m\bar{\psi}\psi + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi + \frac{1}{3}n_f F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{61C_A}{27} - \frac{68n_f}{27}\right) - \frac{4C_F^2}{27}\right)m\bar{\psi}\psi + \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54}\right)F^2\right]$$

$$\eta_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{\psi}\psi - \frac{11}{6}C_A F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{812C_A}{27} - \frac{22n_f}{27}\right) + \frac{85C_F^2}{27}\right)m\bar{\psi}\psi + \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54}\right)F^2\right]$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad C_A = N_c\,, \quad C_F = \frac{N_c^2-1}{2N_c}$$

trace anomaly separately for q, g $\overline{\text{MS}}$ scheme

KT, JHEP01 ('19)120

$$\eta_{\mu\nu}T_q^{\mu\nu} = m\bar{\psi}\psi + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi + \frac{1}{3}n_f F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{61C_A}{27}-\frac{68n_f}{27}\right)-\frac{4C_F^2}{27}\right)m\bar{\psi}\psi + \left(\frac{17C_An_f}{27}+\frac{49C_Fn_f}{54}\right)F^2\right]$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^3\Bigg[\Bigg\{n_f\left(\left(\frac{64\zeta(3)}{9}-\frac{8305}{729}\right)C_F^2-\frac{2}{243}(864\zeta(3)+1079)C_AC_F\right)-\frac{8}{729}(972\zeta(3)+143)C_AC_F^2$$

$$+\left(\frac{32\zeta(3)}{9}+\frac{6611}{729}\right)C_A^2C_F-\frac{76}{243}C_Fn_f^2+\frac{8}{729}(648\zeta(3)-125)C_F^3\Bigg\}m\bar{\psi}\psi$$

$$+\Bigg\{n_f\left(\left(\frac{52\zeta(3)}{9}-\frac{401}{324}\right)C_AC_F+\left(\frac{134}{27}-4\zeta(3)\right)C_A^2+\left(\frac{2407}{1458}-\frac{16\zeta(3)}{9}\right)C_F^2\right)+n_f^2\left(-\frac{697C_A}{729}-\frac{169C_F}{1458}\right)\Bigg\}F^2\Bigg]$$

$$\eta_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{\psi}\psi - \frac{11}{6}C_AF^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{812C_A}{27}-\frac{22n_f}{27}\right)+\frac{85C_F^2}{27}\right)m\bar{\psi}\psi + \left(\frac{28C_An_f}{27}-\frac{17C_A^2}{3}+\frac{5C_Fn_f}{54}\right)F^2\right]$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^3\Bigg[\Bigg\{n_f\left(\left(\frac{368\zeta(3)}{9}-\frac{25229}{729}\right)C_F^2-\frac{2}{243}(4968\zeta(3)+1423)C_AC_F\right)+\left(\frac{32\zeta(3)}{3}-\frac{91753}{1458}\right)C_AC_F^2$$

$$+\left(\frac{294929}{1458}-\frac{32\zeta(3)}{9}\right)C_A^2C_F-\frac{554}{243}C_Fn_f^2+\left(\frac{95041}{729}-\frac{64\zeta(3)}{9}\right)C_F^3\Bigg\}m\bar{\psi}\psi$$

$$+\Bigg\{n_f\left(\left(\frac{1123}{162}-\frac{52\zeta(3)}{9}\right)C_AC_F+\left(4\zeta(3)+\frac{293}{36}\right)C_A^2+\frac{16}{729}(81\zeta(3)-98)C_F^2\right)+n_f^2\left(\frac{655C_A}{2916}-\frac{361C_F}{729}\right)-\frac{2857C_A^3}{108}\Bigg\}F^2\Bigg]$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \big(1+\gamma_m(g)\big)m\bar{\psi}\psi \qquad C_A = N_c\,, \quad C_F = \frac{N_c^2-1}{2N_c}$$

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

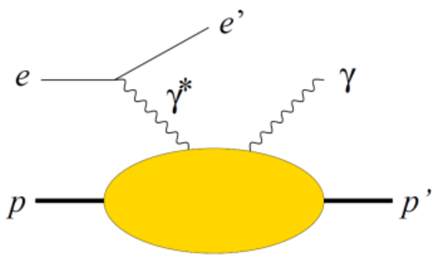
$$\eta_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T^{\mu\nu}_q|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$\left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$



DVCS

$$P = \frac{p+p'}{2}$$

JLab, HERMES, COMPASS, EIC

$$\int \frac{dz^-}{2\pi} e^{ixPz} \langle N(p') | \bar{\psi}(-\frac{z^-}{2}) \gamma^+ \psi(\frac{z^-}{2}) | N(p) \rangle = \frac{1}{P^+} \left[H^q(x,\eta,t) \bar{u}(p') \gamma^+ u(p) + E^q(x,\eta,t) \bar{u}(p') \frac{i\sigma^{+\alpha}(p'-p)_\alpha}{2M} u(p) \right]$$

$$\int \frac{dz^-}{2\pi} e^{ixPz} \langle N(p') | \bar{\psi}(-\frac{z^-}{2}) \gamma^+ \gamma_5 \psi(\frac{z^-}{2}) | N(p) \rangle = \frac{1}{P^+} \left[\tilde{H}^q(x,\eta,t) \bar{u}(p') \gamma^+ \gamma_5 u(p) + \tilde{E}^q(x,\eta,t) \bar{u}(p') \frac{\gamma_5 (p'-p)^+}{2M} u(p) \right]$$

$$\Delta^\mu = p'^\mu - p^\mu \to 0$$

$$\left(t = \Delta^2 \to 0, \quad \eta = \frac{-\Delta \cdot n}{2\bar{P} \cdot n} \to 0 \right)$$

$$H^q(x,0,0) = q(x)$$

$$\int_{-1}^{1} dx H^q(x,\eta,t) = F_1^q(t), \qquad \int_{-1}^{1} dx E^q(x,\eta,t) = F_2^q(t)$$

$$\int_{-1}^{1} dx\, x H^q(x,\eta,t) = A_q(t) + 4\eta^2 D_q(t), \qquad \int_{-1}^{1} dx\, x E^q(x,\eta,t) = B_q(t) - 4\eta^2 D_q(t)$$

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

$$\eta_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0)+4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0)+\frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T^{\mu\nu}_q|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu}+\frac{\eta^{\mu\nu}}{4}F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad \left(\beta(g)\equiv\mu\frac{dg}{d\mu},\ \gamma_m(g)=-\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$$\langle N(p)|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

$$\eta_{\mu\nu}\langle N(p)|T_{q,g}^{\mu\nu}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T_q^{\mu\nu}|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2} \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad \left(\beta(g)\equiv\mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$$\int_0^1 dx\,x\left[q(x,\mu)+\bar{q}(x,\mu)\right] = A_q(t=0,\mu)$$

$$A_q(0,\mu) = \frac{n_f}{4C_F+n_f} + \frac{4C_F A_q(0,\mu_0) + n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}} + \cdots$$

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right)=-\frac{1}{4}A_q(0)+\frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T_q^{\mu\nu}|N(p)\rangle$$

$$A_q(0,\mu)=\int_0^1 dxx[q(x,\mu)+\bar{q}(x,\mu)]$$
$$=\frac{n_f}{4C_F+n_f}+\frac{4C_F A_q(0,\mu_0)+n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}+\cdots$$

$$\eta_{\mu\nu}T_q^{\mu\nu}=m\bar{\psi}\psi+\frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi+\frac{1}{3}n_f F^2\right)+\cdots$$

$$\eta_{\mu\nu}T^{\mu\nu}=\frac{\beta(g)}{2g}F^2+\left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle=2p^\mu p^\nu$$

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right)=-\frac{1}{4}A_q(0)+\frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T_q^{\mu\nu}|N(p)\rangle$$

$$A_q(0,\mu)=\int_0^1 dx\,x\left[q(x,\mu)+\bar{q}(x,\mu)\right]$$
$$=\frac{n_f}{4C_F+n_f}+\frac{4C_F A_q(0,\mu_0)+n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}+\cdots$$

$$\eta_{\mu\nu}T_q^{\mu\nu}=m\bar{\psi}\psi+\frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi+\frac{1}{3}n_f F^2\right)+\cdots$$

$$\eta_{\mu\nu}T^{\mu\nu}=\frac{\beta(g)}{2g}F^2+\left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle=2p^\mu p^\nu$$

$$2M^2=\frac{\beta(g)}{2g}\langle N(p)|F^2|N(p)\rangle+\left(1+\gamma_m(g)\right)\langle N(p)|m\bar{\psi}\psi|N(p)\rangle$$

$$
\bar{C}_q(0,\mu) = -\frac{1}{4}\left(\frac{n_f}{4C_F+n_f}+\frac{2n_f}{3\beta_0}\right)+\frac{1}{4}\left(\frac{2n_f}{3\beta_0}+1\right)\frac{\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle}{2M^2}
$$

$$
-\frac{4C_F A_q(\mu_0)+n_f\left(A_q(\mu_0)-1\right)}{4(4C_F+n_f)}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}
$$

$$
+\frac{\alpha_s(\mu)}{4\pi}\left[\frac{n_f}{4\beta_0}\left(-\frac{34C_A}{27}-\frac{49C_F}{27}\right)+\frac{\beta_1 n_f}{6\beta_0^2}\right] -
$$

$$
+\left[\frac{n_f\,(34C_A+157C_F)}{108\beta_0}+\frac{C_F}{3}-\frac{\beta_1 n_f}{6\beta_0^2}\right]\frac{\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle}{2M^2}\Bigg)-\frac{1}{4}A_q^{\mathrm{NLO}}(\mu)
$$

$$
+\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\Bigg(\frac{n_f^2}{\beta_0}\left[\frac{697C_A}{1458}+\frac{169C_F}{2916}\right]+n_f\left[\frac{17\beta_1 C_A}{54\beta_0^2}+\frac{\beta_2}{6\beta_0^2}+\frac{49\beta_1 C_F}{108\beta_0^2}\right.
$$

$$
\left.+\frac{1}{\beta_0}\left\{\left(\frac{401}{648}-\frac{26\zeta(3)}{9}\right)C_AC_F+\left(2\zeta(3)-\frac{67}{27}\right)C_A^2+\left(\frac{8\zeta(3)}{9}-\frac{2407}{2916}\right)C_F^2\right\}-\frac{\beta_1^2}{6\beta_0^3}\right]
$$

$$
+\Bigg[-\frac{n_f^2}{\beta_0}\left(\frac{697C_A}{1458}+\frac{1789C_F}{2916}\right)+n_f\left(-\frac{17\beta_1 C_A}{54\beta_0^2}-\frac{\beta_2}{6\beta_0^2}-\frac{157\beta_1 C_F}{108\beta_0^2}+\frac{\beta_1^2}{6\beta_0^3}-\frac{17C_F}{27}\right)
$$

$$
+\frac{n_f}{\beta_0}\left\{\left(\frac{26\zeta(3)}{9}+\frac{4315}{648}\right)C_AC_F+\left(\frac{67}{27}-2\zeta(3)\right)C_A^2+\left(\frac{11803}{2916}-\frac{8\zeta(3)}{9}\right)C_F^2\right\}
$$

$$
+\frac{61C_AC_F}{108}-\frac{C_F^2}{27}\Bigg]\frac{\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle}{2M^2}\Bigg)-\frac{1}{4}A_q^{\mathrm{NNLO}}(\mu)\ ,
$$

$$\bar{C}_q(0,\mu) = -\frac{1}{4}\left(\frac{n_f}{4C_F+n_f}+\frac{2n_f}{3\beta_0}\right)+\frac{1}{4}\left(\frac{2n_f}{3\beta_0}+1\right)\frac{\langle N(p)|\,m\bar{\psi}\psi\,|N(p)\rangle}{2M^2}$$

$$-\frac{4C_F A_q(\mu_0)+n_f\left(A_q(\mu_0)-1\right)}{4(4C_F+n_f)}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}$$

$$+\frac{\alpha_s(\mu)}{4\pi}\left[\frac{n_f}{4\beta_0}\left(-\frac{34C_A}{27}-\frac{49C_F}{27}\right)+\frac{\beta_1 n_f}{6\beta_0^2}\right]-$$

$$+\left[\frac{n_f(34C_A+157C_F)}{108\beta_0}+\frac{C_F}{3}-\frac{\beta_1 n_f}{6\beta_0^2}\right]\left.\frac{\langle N(p)|\,m\bar{\psi}\psi\,|N(p)\rangle}{2M^2}\right)-\frac{1}{4}A_q^{\mathrm{NLO}}(\mu)$$

$$+\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left(\frac{n_f^2}{\beta_0}\left[\frac{697C_A}{1458}+\frac{169C_F}{2916}\right]+n_f\left[\frac{17\beta_1C_A}{54\beta_0^2}+\frac{\beta_2}{6\beta_0^2}+\frac{49\beta_1C_F}{108\beta_0^2}\right.\right.$$

$$\left.+\frac{1}{\beta_0}\left\{\left(\frac{401}{648}-\frac{26\zeta(3)}{9}\right)C_AC_F+\left(2\zeta(3)-\frac{67}{27}\right)C_A^2+\left(\frac{8\zeta(3)}{9}-\frac{2407}{2916}\right)C_F^2\right\}-\frac{\beta_1^2}{6\beta_0^3}\right]$$

$$+\left[-\frac{n_f^2}{\beta_0}\left(\frac{697C_A}{1458}+\frac{1789C_F}{2916}\right)+n_f\left(-\frac{17\beta_1C_A}{54\beta_0^2}-\frac{\beta_2}{6\beta_0^2}-\frac{157\beta_1C_F}{108\beta_0^2}+\frac{\beta_1^2}{6\beta_0^3}-\frac{17C_F}{27}\right)\right.$$

$$+\frac{n_f}{\beta_0}\left\{\left(\frac{26\zeta(3)}{9}+\frac{4315}{648}\right)C_AC_F+\left(\frac{67}{27}-2\zeta(3)\right)C_A^2+\left(\frac{11803}{2916}-\frac{8\zeta(3)}{9}\right)C_F^2\right\}$$

$$\left.\left.+\frac{61C_AC_F}{108}-\frac{C_F^2}{27}\right]\frac{\langle N(p)|\,m\bar{\psi}\psi\,|N(p)\rangle}{2M^2}\right)-\frac{1}{4}A_q^{\mathrm{NNLO}}(\mu)\,,$$

$A_q(\mu_0)$

$\langle N(p)|\,m\bar{\psi}\psi\,|N(p)\rangle$

$$A_q(\mu_0) = \int_0^1 dx x\left[q(x,\mu_0) + \overline{q}(x,\mu_0)\right]$$ global QCD analysis at NNLO

$$A_q\left(\mu_0 = 1.3\text{GeV}\right) = 0.613$$

CT18

(MMHT2014,NNPDF)

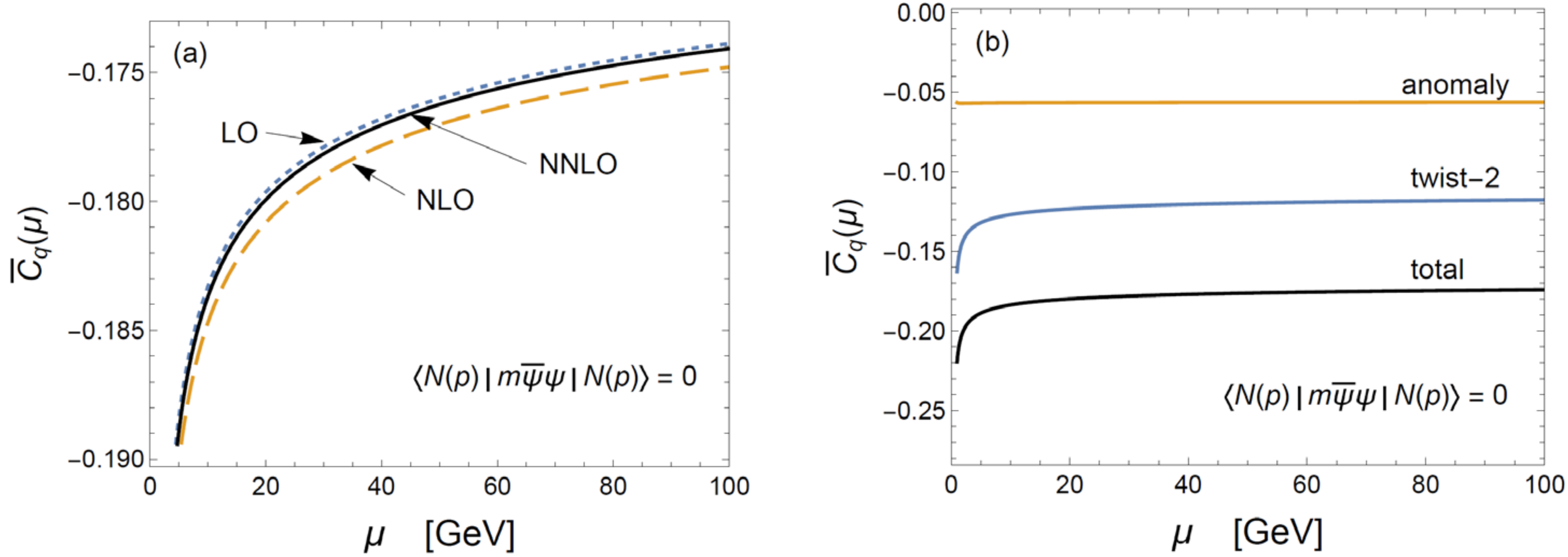
$$\langle N(p)|m\overline{\psi}\psi|N(p)\rangle = \langle N(p)|m_u\overline{u}u + m_d\overline{d}d + m_s\overline{s}s|N(p)\rangle = 2M\left(\sigma_{\pi N} + \sigma_s\right)$$

$$\sigma_{\pi N} = \frac{1}{2M}\langle N(p)|\frac{m_u + m_d}{2}\left(\overline{u}u + \overline{d}d\right)|N(p)\rangle = 59.1 \pm 3.5\,\text{MeV}$$

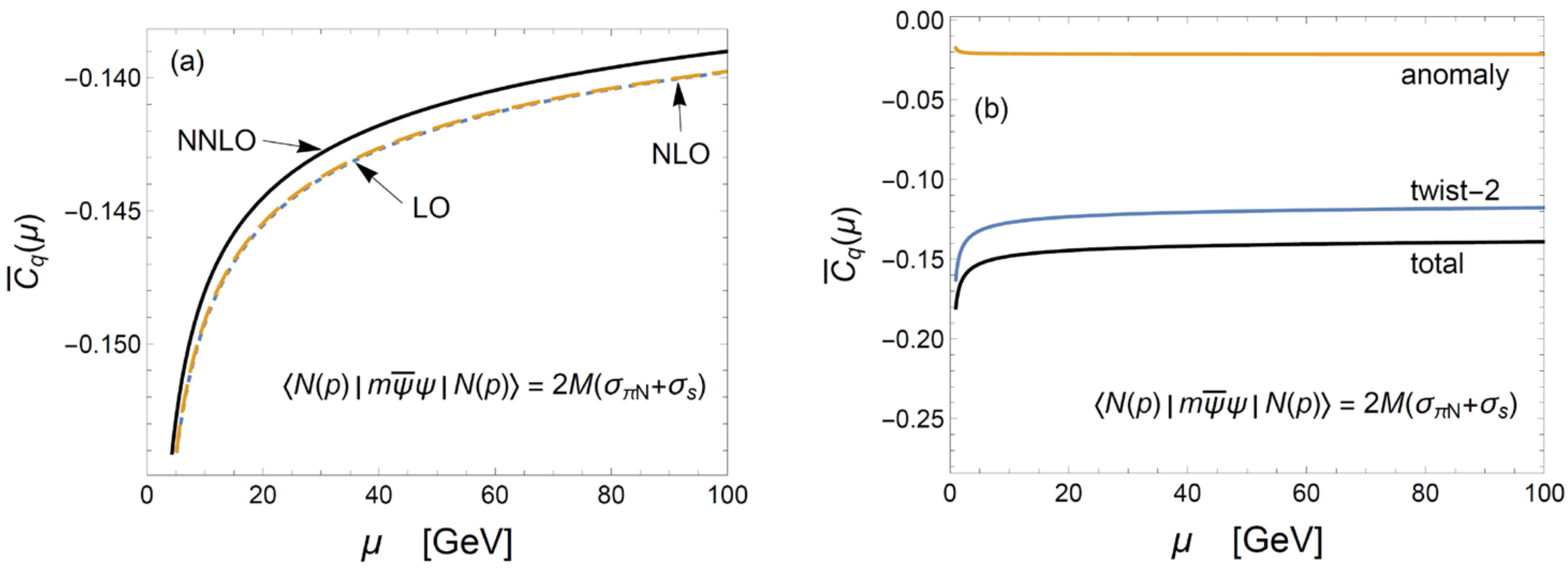
Hoferichter, Elvira, Kubis, Meißner, PRL115, 092301

$$\sigma_s = \frac{1}{2M}\langle N(p)|m_s\overline{s}s|N(p)\rangle = 45.6 \pm 6.2\,\text{MeV}$$

Alexandrou, et al., PRD102, 054517



(a)
LO
NNLO
NLO
⟨N(p) | m$\overline{\psi}\psi$ | N(p)⟩ = 0
$\overline{C}_q(\mu)$
μ [GeV]
(b)
anomaly
twist−2
total
⟨N(p) | m$\overline{\psi}\psi$ | N(p)⟩ = 0
$\overline{C}_q(\mu)$
μ [GeV]



(a)
NNLO
LO
NLO
⟨N(p) | m$\overline{\psi}\psi$ | N(p)⟩ = 2M($\sigma_{\pi N}$+σ_s)
$\overline{C}_q(\mu)$
μ [GeV]
(b)
anomaly
twist−2
total
⟨N(p) | m$\overline{\psi}\psi$ | N(p)⟩ = 2M($\sigma_{\pi N}$+σ_s)
$\overline{C}_q(\mu)$
μ [GeV]

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right)=-\frac{1}{4}A_q(0)+\frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T_q^{\mu\nu}|N(p)\rangle$$

$$A_q(0,\mu)=\int_0^1 dxx\left[q(x,\mu)+\bar{q}(x,\mu)\right]$$

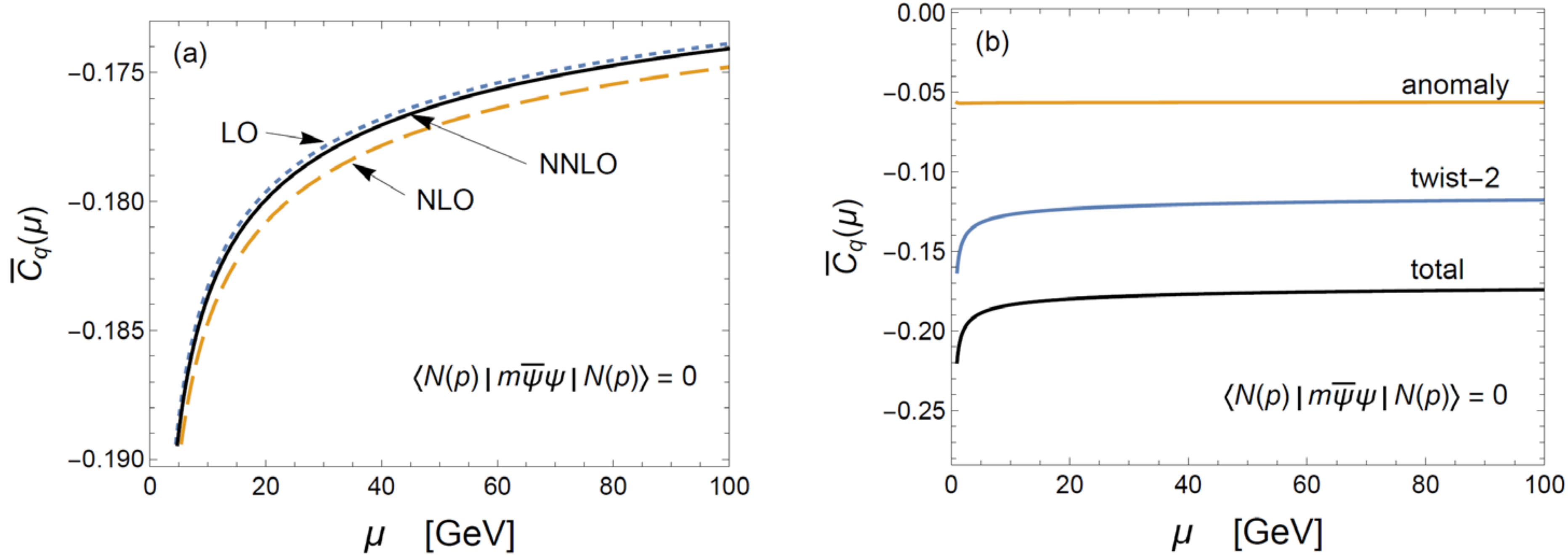
$$=\frac{n_f}{4C_F+n_f}+\frac{4C_F A_q(0,\mu_0)+n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}+\cdots$$

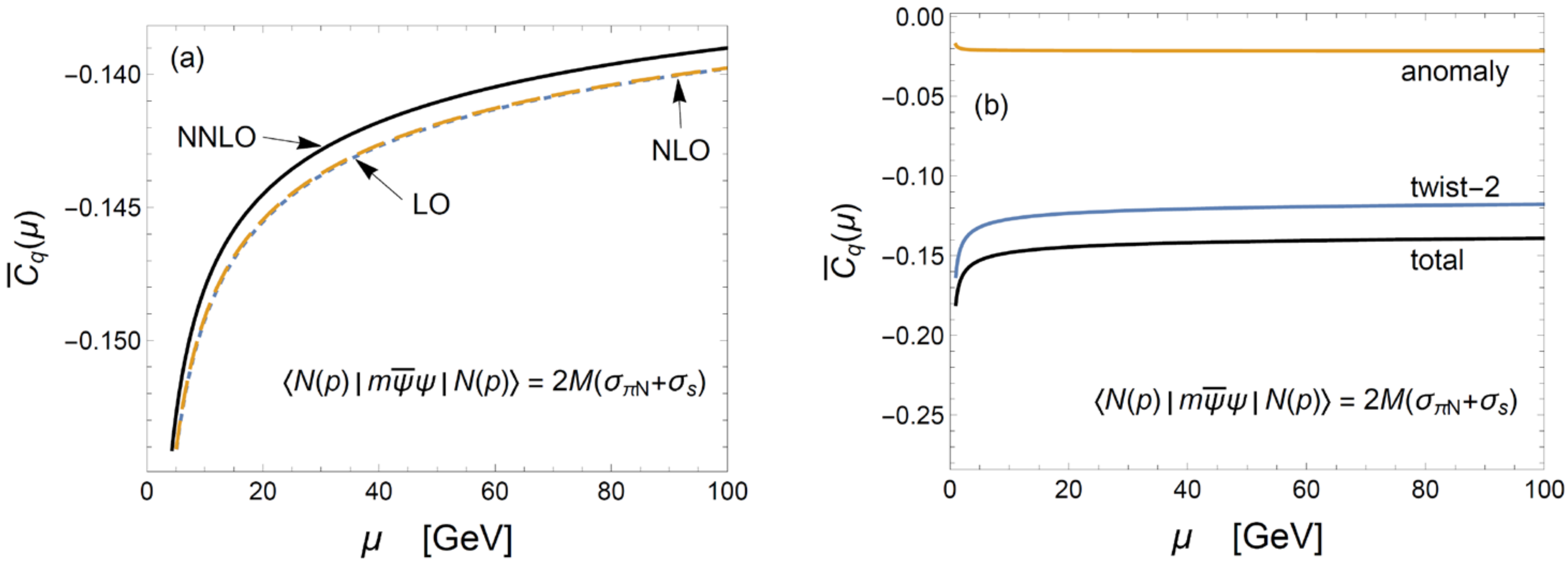
$$\eta_{\mu\nu}T_q^{\mu\nu}=m\bar{\psi}\psi+\frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi+\frac{1}{3}n_f F^2\right)+\cdots$$

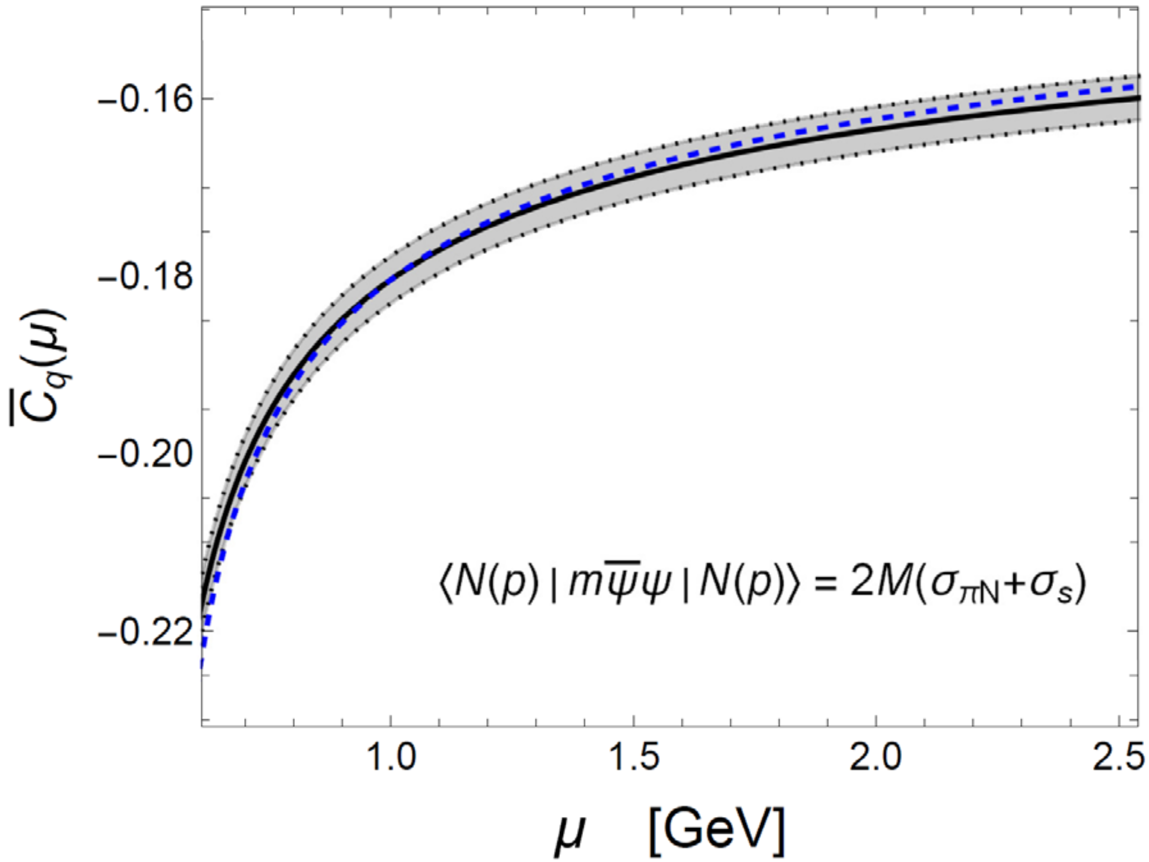
$$\eta_{\mu\nu}T^{\mu\nu}=\frac{\beta(g)}{2g}F^2+\left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle=2p^\mu p^\nu$$

$$2M^2=\frac{\beta(g)}{2g}\langle N(p)|F^2|N(p)\rangle+\left(1+\gamma_m(g)\right)\langle N(p)|m\bar{\psi}\psi|N(p)\rangle$$


(a)
LO
NNLO
NLO
$\langle N(p)\,|\,m\overline{\psi}\psi\,|\,N(p)\rangle = 0$
$\overline{C}_q(\mu)$
μ [GeV]
(b)
anomaly
twist−2
total
$\langle N(p)\,|\,m\overline{\psi}\psi\,|\,N(p)\rangle = 0$
$\overline{C}_q(\mu)$
μ [GeV]


(a)
NNLO
LO
NLO
$\langle N(p)\,|\,m\overline{\psi}\psi\,|\,N(p)\rangle = 2M(\sigma_{\pi N}+\sigma_s)$
$\overline{C}_q(\mu)$
μ [GeV]
(b)
anomaly
twist−2
total
$\langle N(p)\,|\,m\overline{\psi}\psi\,|\,N(p)\rangle = 2M(\sigma_{\pi N}+\sigma_s)$
$\overline{C}_q(\mu)$
μ [GeV]



$$\bar{C}_q(\mu = 0.7\ \mathrm{GeV})\big|_{n_f=3} = -0.201 \pm 0.003$$

$$\bar{C}_q(\mu = 1\ \mathrm{GeV})\big|_{n_f=3} = -0.180 \pm 0.003$$

$$\bar{C}_q(\mu = 2\ \mathrm{GeV})\big|_{n_f=3} = -0.163 \pm 0.003$$

$\overline{\mathrm{MS}}$ scheme

$$\bar{C}_q(\mu)\big|_{n_f=3} \simeq -0.108 - 0.114\left[\alpha_s\left(\mu\right)\right]^{\frac{50}{81}}$$

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right)=-\frac{1}{4}A_q(0)+\frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T_q^{\mu\nu}|N(p)\rangle$$

$$A_q(0,\mu)=\int_0^1 dxx\left[q(x,\mu)+\bar{q}(x,\mu)\right]$$
$$=\frac{n_f}{4C_F+n_f}+\frac{4C_F A_q(0,\mu_0)+n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}+\cdots$$

$$\eta_{\mu\nu}T_q^{\mu\nu}=m\bar{\psi}\psi+\frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi+\frac{1}{3}n_f F^2\right)+\cdots$$

$$\eta_{\mu\nu}T^{\mu\nu}=\frac{\beta(g)}{2g}F^2+\left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle=2p^\mu p^\nu$$

$$2M^2=\frac{\beta(g)}{2g}\langle N(p)|F^2|N(p)\rangle+\left(1+\gamma_m(g)\right)\langle N(p)|m\bar{\psi}\psi|N(p)\rangle$$

$$A_q(\mu_0) = \int_0^1 dx\, x\left[q(x,\mu_0) + \overline{q}(x,\mu_0)\right]$$

global QCD analysis at NNLO

$$A_q\left(\mu_0 = 1.3\,\mathrm{GeV}\right) = 0.613$$

CT18

(MMHT2014, NNPDF)

$$\langle N(p)|\, m\overline{\psi}\psi\, |N(p)\rangle = \langle N(p)|\, m_u\overline{u}u + m_d\overline{d}d + m_s\overline{s}s\, |N(p)\rangle = 2M\left(\sigma_{\pi N} + \sigma_s\right)$$

$$\sigma_{\pi N} = \frac{1}{2M}\langle N(p)|\,\frac{m_u + m_d}{2}\left(\overline{u}u + \overline{d}d\right)|N(p)\rangle = 59.1 \pm 3.5\,\mathrm{MeV}$$

Hoferichter, Elvira, Kubis, Meißner, PRL115, 092301

$$\sigma_s = \frac{1}{2M}\langle N(p)|\, m_s\overline{s}s\, |N(p)\rangle = 45.6 \pm 6.2\,\mathrm{MeV}$$

Alexandrou, et al., PRD102, 054517

$$A_q^\pi(\mu_0) = \int_0^1 dx\, x\left[q^\pi(x,\mu_0) + \bar{q}^\pi(x,\mu_0)\right]$$

global QCD analysis at NLO

$$A_q^\pi\left(\mu_0 = 1.3\,\mathrm{GeV}\right) = \begin{cases} 0.70 \pm 0.02 \\ 0.81 \pm 0.16 \\ 0.61 \pm 0.08 \end{cases}$$

JAM ('18)

xFitter ('20)

JAM ('21)

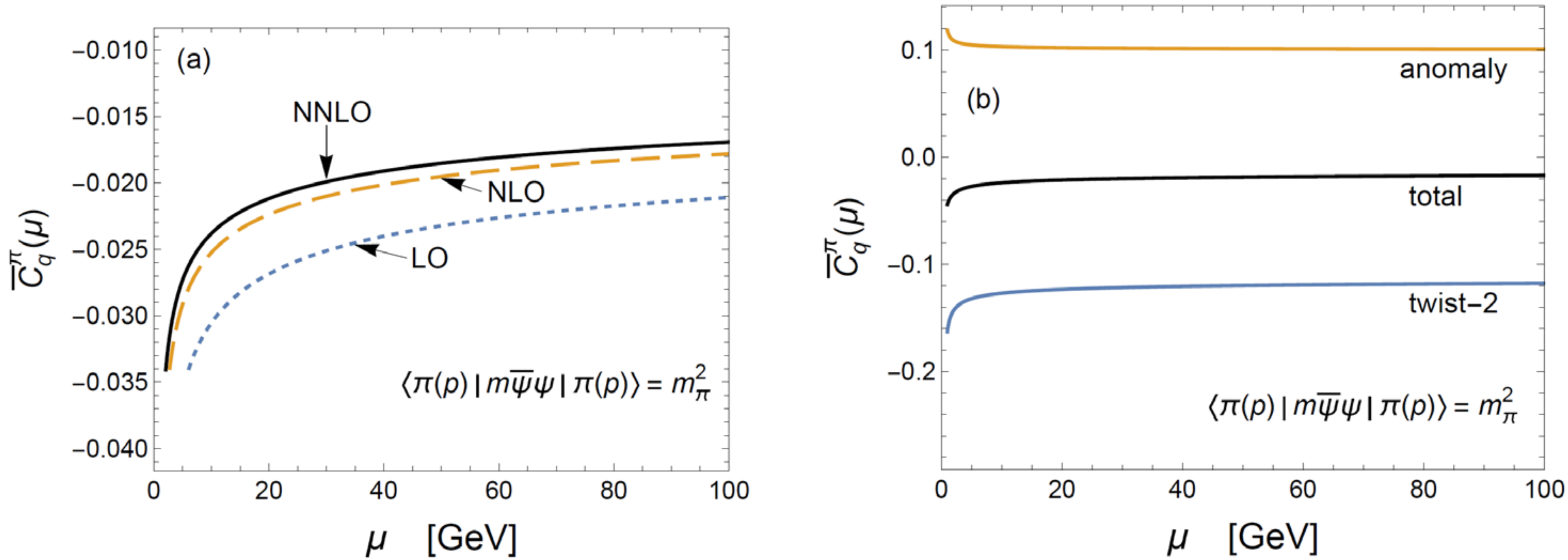
$$\langle \pi(p) | m\bar{\psi}\psi | \pi(p) \rangle$$

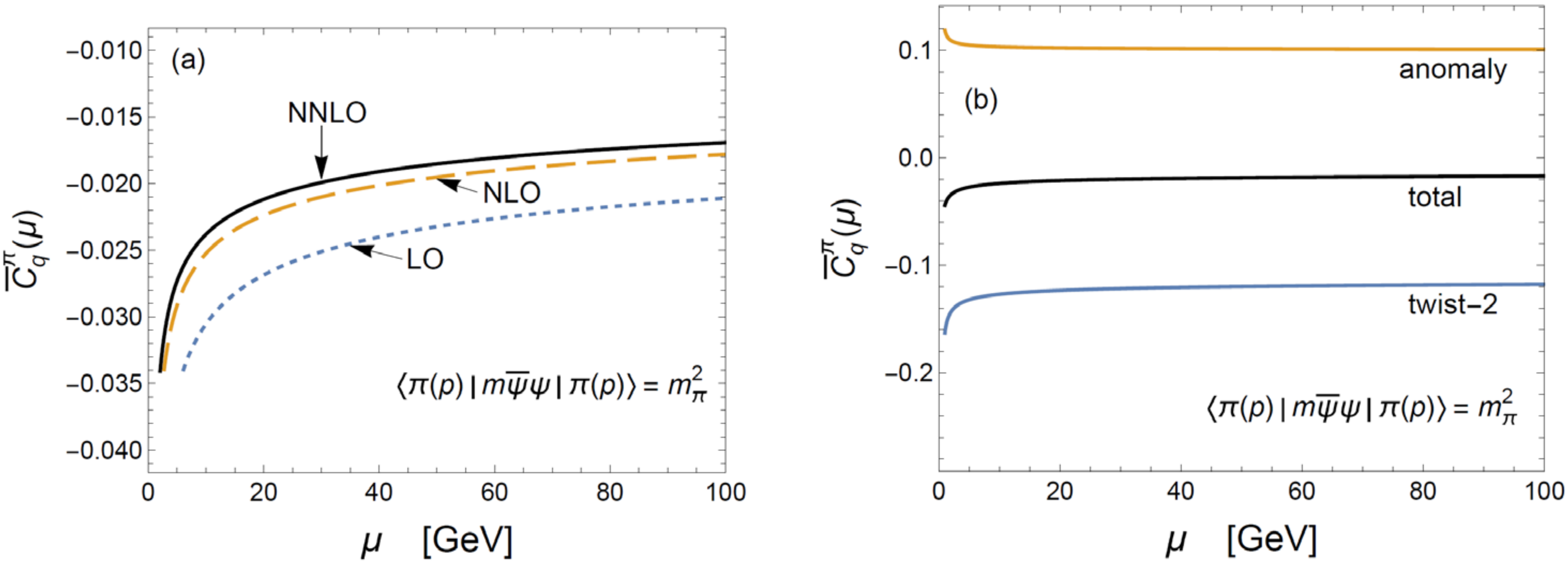
$$= m_\pi^2 + O(6\%)$$

χPT

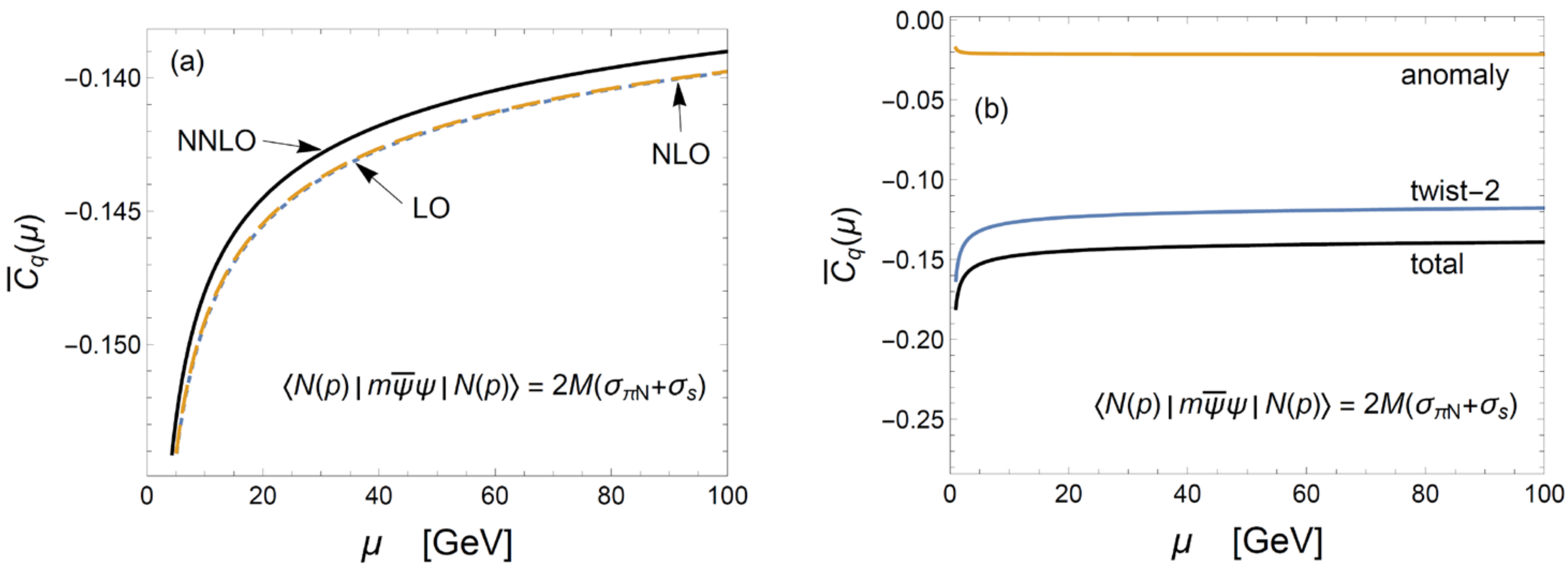
Gasser, Leutwyler, Annals Phys. 158, 142

Colangelo, Gasser, Leutwyler, PRL86, 5008

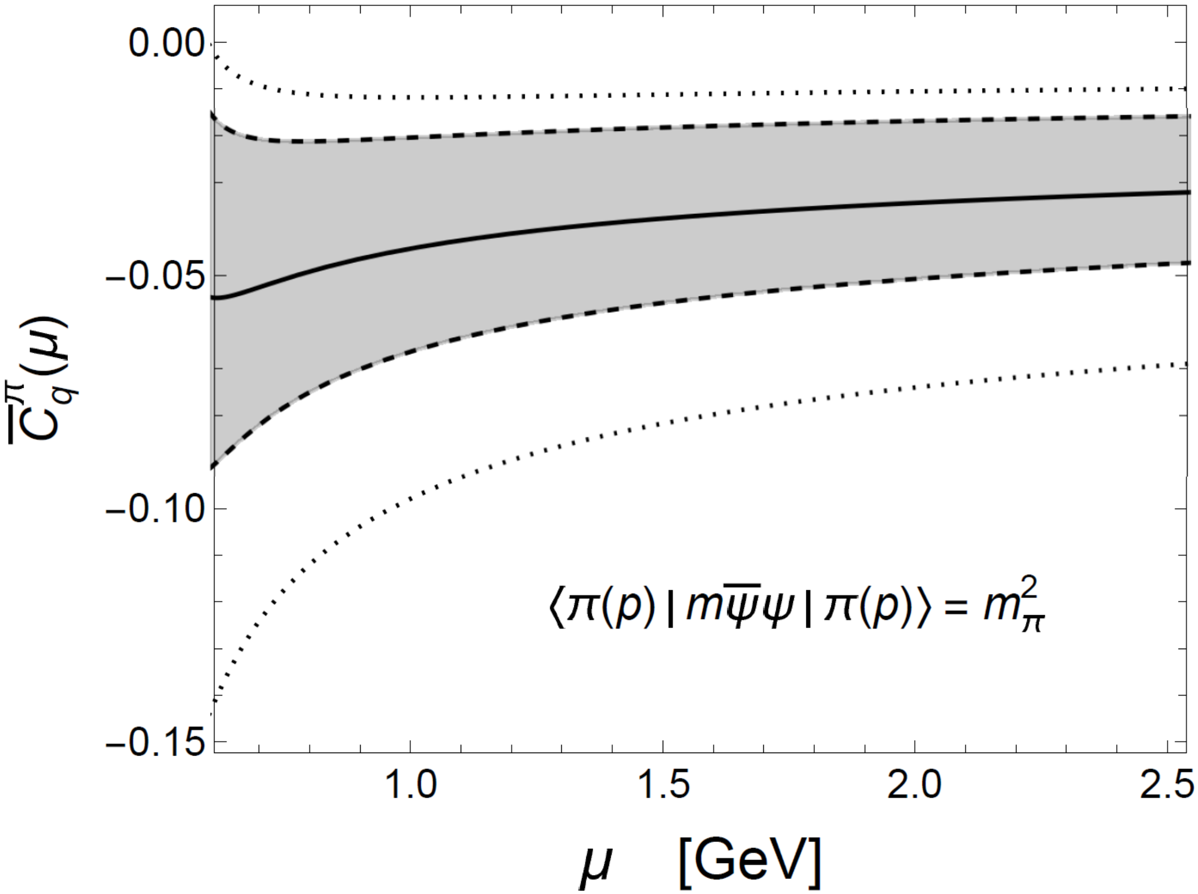

(a)
NNLO
NLO
LO
$\langle \pi(p) | m\overline{\psi}\psi | \pi(p) \rangle = m_\pi^2$
$\overline{C}_q^\pi(\mu)$
μ [GeV]
−0.010
−0.015
−0.020
−0.025
−0.030
−0.035
−0.040
0
20
40
60
80
100
(b)
anomaly
total
twist-2
$\langle \pi(p) | m\overline{\psi}\psi | \pi(p) \rangle = m_\pi^2$
$\overline{C}_q^\pi(\mu)$
μ [GeV]
0.1
0.0
−0.1
−0.2
0
20
40
60
80
100



(a)
NNLO
NLO
LO
$\langle \pi(p) | m\overline{\psi}\psi | \pi(p) \rangle = m_\pi^2$
$\overline{C}_q^\pi(\mu)$
μ [GeV]
(b)
anomaly
total
twist−2
$\langle \pi(p) | m\overline{\psi}\psi | \pi(p) \rangle = m_\pi^2$
$\overline{C}_q^\pi(\mu)$
μ [GeV]



(a)
NNLO
LO
NLO
$\langle N(p) | m\overline{\psi}\psi | N(p) \rangle = 2M(\sigma_{\pi N}+\sigma_s)$
$\overline{C}_q(\mu)$
μ [GeV]
(b)
anomaly
twist−2
total
$\langle N(p) | m\overline{\psi}\psi | N(p) \rangle = 2M(\sigma_{\pi N}+\sigma_s)$
$\overline{C}_q(\mu)$
μ [GeV]



$$\bar{C}_q^\pi(\mu = 0.7\ \text{GeV})\big|_{n_f=3} = -0.05 \pm 0.03$$

$$\bar{C}_q^\pi(\mu = 1\ \text{GeV})\big|_{n_f=3} = -0.04 \pm 0.02$$

$$\bar{C}_q^\pi(\mu = 2\ \text{GeV})\big|_{n_f=3} = -0.03 \pm 0.02$$

$\overline{\text{MS}}$ scheme

at $t = 0$:

$\bar{C}_q(0, \mu \sim 0.4 \text{ GeV}) = 0.25$ **Bag model** [Ji, Melnitchouk, Song, PRD56, 5511 ('97)]

$\bar{C}_q(0, \mu = 2 \text{ GeV}) \approx -0.11$ **Phenomenological** [Lorce, EPJC78, 120 ('18)]

$\bar{C}_q(0, \mu \sim 0.63 \text{ GeV}) = 0.014$ **Instanton** [Polyakov, Son, JHEP09, 156 ('18)]

$\bar{C}_q(0, \mu = 1 \text{ GeV}) = -0.021 \pm 0.008$ **LCSR** [Azizi,Ozdem, EPJC80, 104 ('20)]

$\bar{C}_q(0, \mu \to \infty) \simeq -0.15$ **Trace anomaly** [Hatta, Rajan, KT, JHEP12, 008 ('18)]

nucleon

$\bar{C}_q(0, \mu \sim 0.4\ \mathrm{GeV}) = 0.25$ **Bag model** [Ji, Melnitchouk, Song, PRD56, 5511 ('97)]

$\bar{C}_q(0, \mu = 2\ \mathrm{GeV}) \approx -0.11$ **Phenomenological** [Lorce, EPJC78, 120 ('18)]

$\bar{C}_q(0, \mu \sim 0.63\ \mathrm{GeV}) = 0.014$ **Instanton** [Polyakov, Son, JHEP09, 156 ('18)]

$\bar{C}_q(0, \mu = 1\ \mathrm{GeV}) = -0.021 \pm 0.008$ **LCSR** [Azizi,Ozdem, EPJC80, 104 ('20)]

$\bar{C}_q(0, \mu \to \infty) \simeq -0.15$ **Trace anomaly** [Hatta, Rajan, KT, JHEP12, 008 ('18)]

$\bar{C}_q(0, \mu = 1\ \mathrm{GeV}) = -0.180 \pm 0.003$

$\bar{C}_q(0, \mu = 2\ \mathrm{GeV}) = -0.163 \pm 0.003$

NNLO QCD [KT, JHEP03, 013 ('23)]

pion

$\bar{C}_q^{\pi}(0, \mu = 1\ \mathrm{GeV}) = -0.04 \pm 0.02$ **NNLO QCD with NLO input** [KT, JHEP03, 013 ('23)]

① mass decomposition

Ji, PRD52 271 ('95)

Lorce, Moutarde, Trawinski, EPJC79, 89 ('19)

Metz, Pasquini, Rodini, PRD102, 114042 ('20)

Ji, Liu, Schafer, NPB971, 115537 ('21)

② pressure

$$-\bar{C}_{q,g}\frac{M}{V}$$

Lorce, EPJC78, 120 ('18)

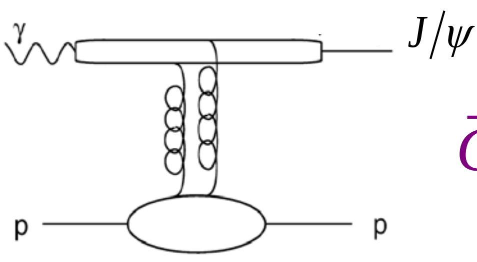
Liu, PRD104, 076010 ('21)

③ nucleon's transverse spin sum rule

$$J_{q,g}=\frac{1}{2}(A_{q,g}+B_{q,g})+\frac{p^3}{2(p^0+M)}\bar{C}_{q,g}$$

Hatta, KT, Yoshida, JHEP 02 ('13) 003

④ $\gamma p \to J/\psi\ p$ near threshold JLab, EIC



$$\bar{C}_g\ \left(=-\bar{C}_q\right)$$

Y. Hatta, D. Yang, PRD98, 074003

Y. Hatta, A. Rajan, D. Yang, PRD100, 014032

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

$$\eta_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|\eta_{\mu\nu}T^{\mu\nu}_q|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_\rho{}^\nu + \frac{\eta^{\mu\nu}}{4}F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$\eta_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad \left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$$\int_0^1 dx\, x\left[q(x,\mu) + \bar{q}(x,\mu)\right] = A_q(t=0,\mu)$$

$$A_q(0,\mu) = \frac{n_f}{4C_F + n_f} + \frac{4C_F A_q(0,\mu_0) + n_f\left(A_q(0,\mu_0) - 1\right)}{4C_F + n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F + 2n_f}{3\beta_0}} + \cdots$$

$$\langle N(p)|T^{00}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(0}p^{0)} + \bar{C}_{q,g}(0)M\eta^{00}\right]u(p)$$

$$\left\langle \hat{H}_{q,g} \right\rangle = \frac{\langle N|\int d^3x T^{00}_{q,g}|N\rangle}{\langle N|N\rangle} = M\left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \int d^3x \psi^\dagger\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha} + m\beta\right)\psi + \int d^3x \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)$$

$$M = M_q + M_g$$

$$M_{q,g} = \left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right)M$$

$$\langle N(p)|T^{00}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(0}p^{0)} + \bar{C}_{q,g}(0)M\eta^{00}\right]u(p)$$

$$\left\langle \hat{H}_{q,g} \right\rangle = \frac{\langle N|\int d^3x T^{00}_{q,g}|N\rangle}{\langle N|N\rangle} = M\left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x \psi^\dagger\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha} + m\beta\right)\psi} + \boxed{\int d^3x \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)}$$

$$M = M_q + M_g \qquad\qquad M_{q,g} = \left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right)M$$

$$M = \left(M_q - M_m\right) + M_g + M_m \qquad\qquad M_m = \left\langle \int d^3x\, m\bar{\psi}\psi \right\rangle = \frac{\sigma_{\pi N} + \sigma_s}{M} M$$

$$M = \left(M_q - M_m - \Delta M_q\right) + \left(M_g - \Delta M_g\right) + M_m + \left(\Delta M_q + \Delta M_g\right)$$

$$\Delta M_q + \Delta M_g = \frac{1}{4}\left\langle \int d^3x \left(\frac{\beta(g)}{2g}F^2 + \gamma_m(g)m\bar{\psi}\psi\right)\right\rangle = \frac{1}{4}\left(M - M_m\right)$$

$$\langle N(p)|T^{00}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(0}p^{0)} + \bar{C}_{q,g}(0)M\eta^{00}\right]u(p)$$

$$\left\langle \hat{H}_{q,g}\right\rangle = \frac{\langle N|\int d^3x T^{00}_{q,g}|N\rangle}{\langle N|N\rangle} = M\left(A_{q,g}(0)+\bar{C}_{q,g}(0)\right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x \psi^\dagger\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha}+m\beta\right)\psi} + \boxed{\int d^3x \frac{1}{2}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)}$$

$$M = M_q + M_g \qquad M_{q,g} = \left(A_{q,g}(0)+\bar{C}_{q,g}(0)\right)M$$

0.6, 0.4

$$M = \left(M_q - M_m\right) + M_g + M_m \qquad M_m = \left\langle \int d^3x\, m\bar{\psi}\psi\right\rangle = \frac{\sigma_{\pi N}+\sigma_s}{M}M$$

0.1

$$M = \left(M_q - M_m - \Delta M_q\right) + \left(M_g - \Delta M_g\right) + M_m + \left(\Delta M_q + \Delta M_g\right)$$

$$\Delta M_q + \Delta M_g = \frac{1}{4}\left\langle \int d^3x\left(\frac{\beta(g)}{2g}F^2 + \gamma_m(g)m\bar{\psi}\psi\right)\right\rangle = \frac{1}{4}\left(M - M_m\right)$$

0.2

$$\langle N(p)|T^{00}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(0}p^{0)} + \bar{C}_{q,g}(0)M\eta^{00}\right]u(p)$$

$$\left\langle \hat{H}_{q,g} \right\rangle = \frac{\langle N|\int d^3x T^{00}_{q,g}|N\rangle}{\langle N|N\rangle} = M\left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x \psi^\dagger\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha} + m\beta\right)\psi} + \boxed{\int d^3x \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)}$$

$$M = M_q + M_g \qquad M_{q,g} = \left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right)M$$

0.6, 0.4   –0.18, 0.18

$$M = \left(M_q - M_m\right) + M_g + M_m \qquad M_m = \left\langle \int d^3x\, m\bar{\psi}\psi \right\rangle = \frac{\sigma_{\pi N} + \sigma_s}{M}M$$

0.1

$$M = \left(M_q - M_m - \Delta M_q\right) + \left(M_g - \Delta M_g\right) + M_m + \left(\Delta M_q + \Delta M_g\right)$$

$$\Delta M_q + \Delta M_g = \frac{1}{4}\left\langle \int d^3x \left(\frac{\beta(g)}{2g}F^2 + \gamma_m(g)m\bar{\psi}\psi\right)\right\rangle = \frac{1}{4}\left(M - M_m\right)$$

0.2

$$\langle N(p)|T^{00}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(0}p^{0)} + \bar{C}_{q,g}(0)M\eta^{00}\right]u(p)$$

$$\left\langle \hat{H}_{q,g}\right\rangle = \frac{\langle N|\int d^3x T^{00}_{q,g}|N\rangle}{\langle N|N\rangle} = M\left(A_{q,g}(0)+\bar{C}_{q,g}(0)\right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x\psi^\dagger\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha}+m\beta\right)\psi} + \boxed{\int d^3x\frac{1}{2}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)}$$

$$M = \underbrace{M_q}_{0.4} + \underbrace{M_g}_{0.6}$$

$$M_{q,g} = \left(\underbrace{A_{q,g}(0)}_{0.6,\ 0.4}+\underbrace{\bar{C}_{q,g}(0)}_{-0.18,\ 0.18}\right)M$$

$$M = \left(\underbrace{M_q - M_m}_{0.3}\right) + \underbrace{M_g}_{0.6} + \underbrace{M_m}_{0.1}$$

$$\underbrace{M_m}_{0.1} = \left\langle \int d^3x\ m\bar{\psi}\psi\right\rangle = \frac{\sigma_{\pi N}+\sigma_s}{M}M$$

$$M = \left(M_q - M_m - \Delta M_q\right) + \left(M_g - \Delta M_g\right) + \underbrace{M_m}_{0.1} + \underbrace{\left(\Delta M_q + \Delta M_g\right)}_{0.2}$$

$$\underbrace{\Delta M_q + \Delta M_g}_{0.2} = \frac{1}{4}\left\langle \int d^3x\left(\frac{\beta(g)}{2g}F^2 + \gamma_m(g)m\bar{\psi}\psi\right)\right\rangle = \frac{1}{4}\left(M - M_m\right)$$

$$\langle N(p)|T^{00}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(0}p^{0)} + \bar{C}_{q,g}(0)M\eta^{00}\right]u(p)$$

$$\left\langle \hat{H}_{q,g}\right\rangle = \frac{\langle N|\int d^3x T^{00}_{q,g}|N\rangle}{\langle N|N\rangle} = M\left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x \psi^\dagger\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha} + m\beta\right)\psi} + \boxed{\int d^3x \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)}$$

$$M = M_q + M_g$$

0.4 0.6

$$M_{q,g} = \left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right)M$$

0.6, 0.4 –0.18, 0.18

$$M = \left(M_q - M_m\right) + M_g + M_m$$

0.3 0.6 0.1

$$M_m = \left\langle \int d^3x\, m\bar{\psi}\psi \right\rangle = \frac{\sigma_{\pi N} + \sigma_s}{M}M$$

0.1

$$M = \left(M_q - M_m - \Delta M_q\right) + \left(M_g - \Delta M_g\right) + M_m + \left(\Delta M_q + \Delta M_g\right)$$

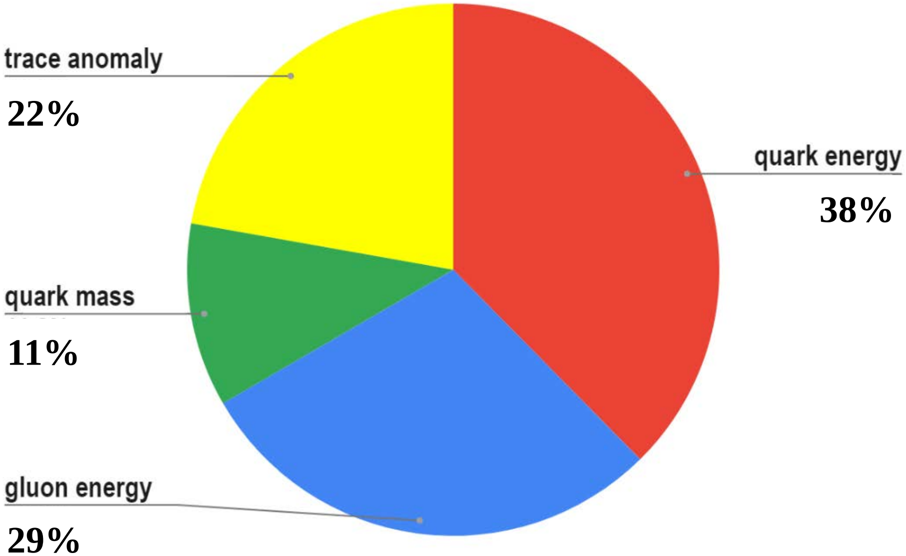
0.4 0.3 0.1 0.2

$$\Delta M_q + \Delta M_g = \frac{1}{4}\left\langle \int d^3x \left(\frac{\beta(g)}{2g}F^2 + \gamma_m(g)m\bar{\psi}\psi\right)\right\rangle = \frac{1}{4}\left(M - M_m\right)$$

0.2

“Ji’s decomposition” in NNLO QCD

proton mass



$$\mu = 1.3 \text{ GeV}$$

$$\langle \pi(p) | T^{00}_{q,g} | \pi(p) \rangle = 2A^{\pi}_{q,g}(0) p^0 p^0 + 2\left(M^{\pi}\right)^2 \bar{C}^{\pi}_{q,g}(0)\eta^{00}$$

$$\left\langle \hat{H}_{q,g} \right\rangle = \frac{\langle \pi | \int d^3x T^{00}_{q,g} | \pi \rangle}{\langle \pi | \pi \rangle} = M^{\pi}\left(A^{\pi}_{q,g}(0) + \bar{C}^{\pi}_{q,g}(0) \right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x \psi^{\dagger}\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha} + m\beta\right)\psi} + \boxed{\int d^3x \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)}$$

$$M^{\pi} = M^{\pi}_q + M^{\pi}_g \qquad M^{\pi}_{q,g} = \left(A^{\pi}_{q,g}(0) + \bar{C}^{\pi}_{q,g}(0) \right) M^{\pi}$$

$$M^{\pi} = \left(M^{\pi}_q - M^{\pi}_m \right) + M^{\pi}_g + M^{\pi}_m \qquad M^{\pi}_m = \left\langle \int d^3x \, m\bar{\psi}\psi \right\rangle$$

$$M^{\pi} = \left(M^{\pi}_q - M^{\pi}_m - \Delta M^{\pi}_q \right) + \left(M^{\pi}_g - \Delta M^{\pi}_g \right) + M^{\pi}_m + \left(\Delta M^{\pi}_q + \Delta M^{\pi}_g \right)$$

$$\Delta M^{\pi}_q + \Delta M^{\pi}_g = \frac{1}{4}\left\langle \int d^3x \left(\frac{\beta(g)}{2g} F^2 + \gamma_m(g) m\bar{\psi}\psi \right) \right\rangle = \frac{1}{4}\left(M^{\pi} - M^{\pi}_m \right)$$

$$\langle \pi(p) | T^{00}_{q,g} | \pi(p) \rangle = 2A^{\pi}_{q,g}(0)\, p^0 p^0 + 2\left(M^{\pi}\right)^2 \bar{C}^{\pi}_{q,g}(0)\, \eta^{00}$$

$$\left\langle \hat{H}_{q,g} \right\rangle = \frac{\langle \pi | \int d^3x T^{00}_{q,g} | \pi \rangle}{\langle \pi | \pi \rangle} = M^{\pi}\left(A^{\pi}_{q,g}(0) + \bar{C}^{\pi}_{q,g}(0)\right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x \psi^{\dagger}\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha} + m\beta\right)\psi} + \boxed{\int d^3x \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)}$$

$$M^{\pi} = M^{\pi}_q + M^{\pi}_g \qquad M^{\pi}_{q,g} = \left(A^{\pi}_{q,g}(0) + \bar{C}^{\pi}_{q,g}(0)\right) M^{\pi}$$

0.6, 0.4

$$M^{\pi} = \left(M^{\pi}_q - M^{\pi}_m\right) + M^{\pi}_g + M^{\pi}_m \qquad M^{\pi}_m = \left\langle \int d^3x\, m\bar{\psi}\psi \right\rangle$$

0.5

$$M^{\pi} = \left(M^{\pi}_q - M^{\pi}_m - \Delta M^{\pi}_q\right) + \left(M^{\pi}_g - \Delta M^{\pi}_g\right) + M^{\pi}_m + \left(\Delta M^{\pi}_q + \Delta M^{\pi}_g\right)$$

$$\Delta M^{\pi}_q + \Delta M^{\pi}_g = \frac{1}{4}\left\langle \int d^3x \left(\frac{\beta(g)}{2g} F^2 + \gamma_m(g)\, m\bar{\psi}\psi\right)\right\rangle = \frac{1}{4}\left(M^{\pi} - M^{\pi}_m\right)$$

0.1

$$\langle \pi(p) | T^{00}_{q,g} | \pi(p) \rangle = 2A^{\pi}_{q,g}(0)\, p^0 p^0 + 2\left(M^{\pi}\right)^2 \bar{C}^{\pi}_{q,g}(0)\eta^{00}$$

$$\left\langle \hat{H}_{q,g} \right\rangle = \frac{\langle \pi | \int d^3x T^{00}_{q,g} | \pi \rangle}{\langle \pi | \pi \rangle} = M^{\pi}\left(A^{\pi}_{q,g}(0) + \bar{C}^{\pi}_{q,g}(0)\right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x \psi^{\dagger}\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha} + m\beta\right)\psi} + \boxed{\int d^3x \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)}$$

$$M^{\pi} = M^{\pi}_q + M^{\pi}_g$$

$$M^{\pi}_{q,g} = \left(A^{\pi}_{q,g}(0) + \bar{C}^{\pi}_{q,g}(0)\right) M^{\pi}$$

0.6, 0.4 –0.04, 0.04

$$M^{\pi} = \left(M^{\pi}_q - M^{\pi}_m\right) + M^{\pi}_g + M^{\pi}_m$$

$$M^{\pi}_m = \left\langle \int d^3x\, m\bar{\psi}\psi \right\rangle$$

0.5

$$M^{\pi} = \left(M^{\pi}_q - M^{\pi}_m - \Delta M^{\pi}_q\right) + \left(M^{\pi}_g - \Delta M^{\pi}_g\right) + M^{\pi}_m + \left(\Delta M^{\pi}_q + \Delta M^{\pi}_g\right)$$

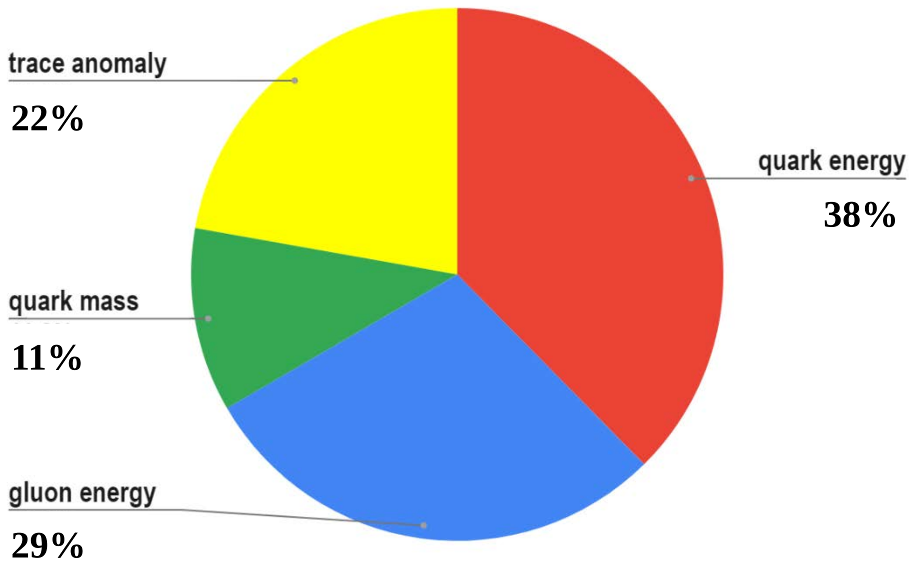
$$\Delta M^{\pi}_q + \Delta M^{\pi}_g = \frac{1}{4}\left\langle \int d^3x \left(\frac{\beta(g)}{2g} F^2 + \gamma_m(g)\, m\bar{\psi}\psi\right) \right\rangle = \frac{1}{4}\left(M^{\pi} - M^{\pi}_m\right)$$

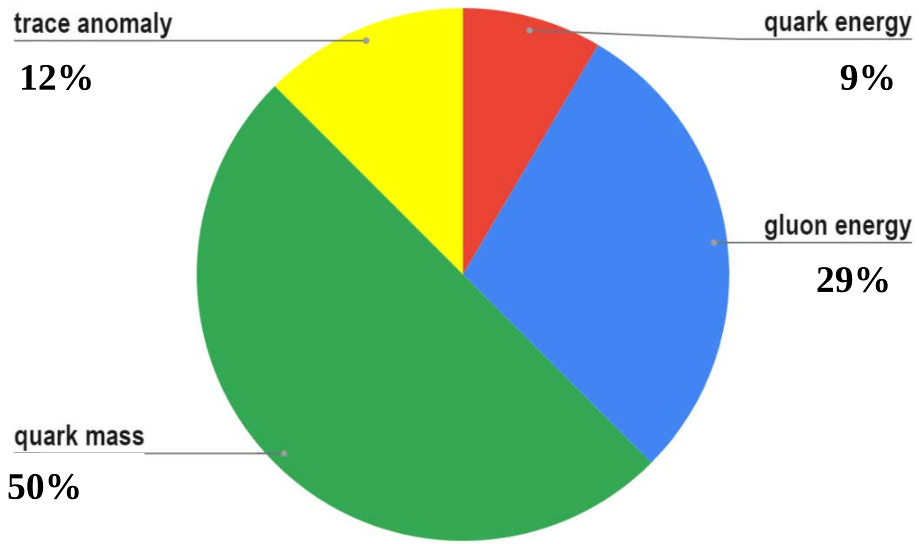
0.1

"Ji's decomposition" in NNLO QCD

proton mass

pion mass

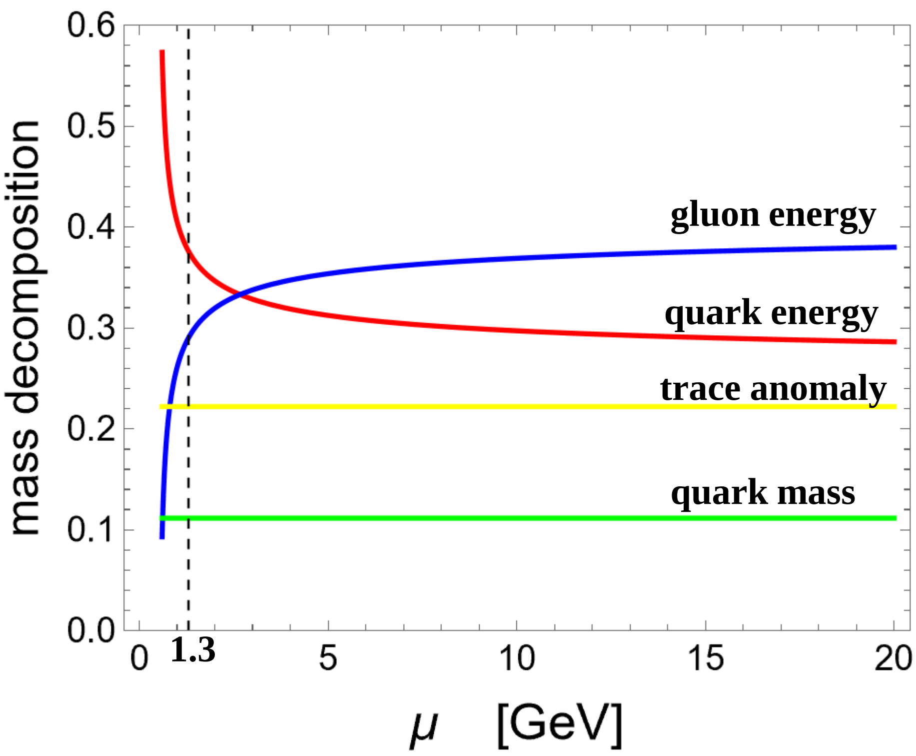




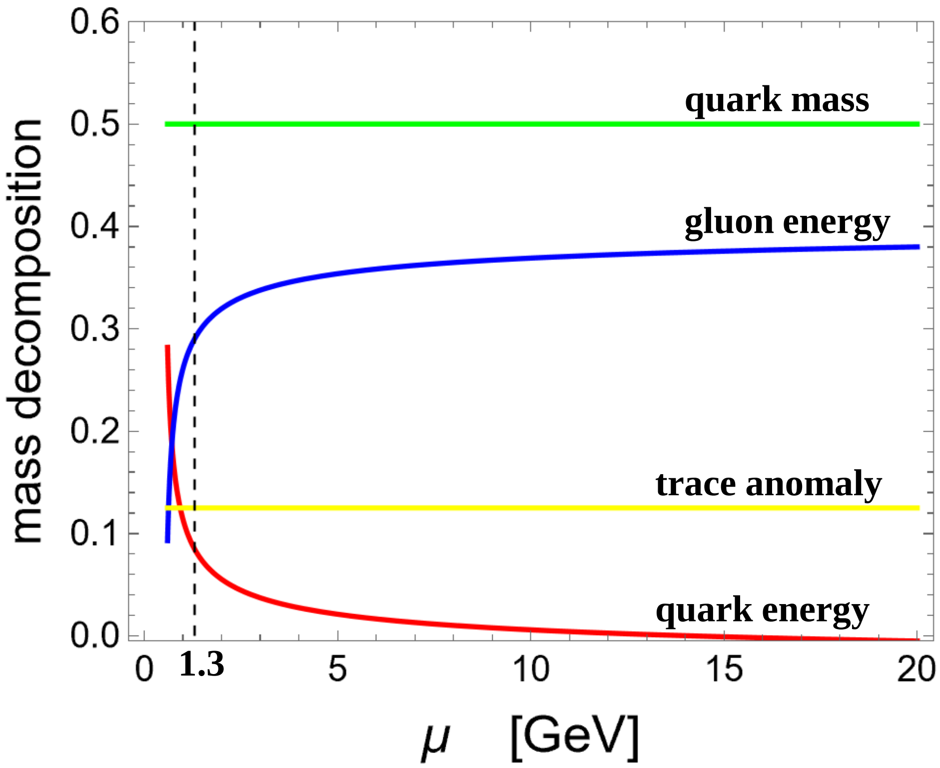
$$\mu = 1.3 \text{ GeV}$$

"Ji's decomposition" in NNLO QCD

proton mass



pion mass



$\overline{\text{MS}}$ scheme

$$\langle N(p) | T^{00}_{q,g} | N(p) \rangle = \bar{u}(p) \left[A_{q,g}(0) \gamma^{(0} p^{0)} + \bar{C}_{q,g}(0) M \eta^{00} \right] u(p)$$

$$\left\langle \hat{H}_{q,g} \right\rangle = \frac{\langle N | \int d^3x T^{00}_{q,g} | N \rangle}{\langle N | N \rangle} = M \left(A_{q,g}(0) + \bar{C}_{q,g}(0) \right)$$

$$\hat{H} = \hat{H}_q + \hat{H}_g = \boxed{\int d^3x \psi^\dagger \left(-i\boldsymbol{D} \cdot \boldsymbol{\alpha} + m\beta \right) \psi} + \boxed{\int d^3x \frac{1}{2} \left(\boldsymbol{E}^2 + \boldsymbol{B}^2 \right)}$$

$$M = M_q + M_g \qquad M_{q,g} = \left(A_{q,g}(0) + \bar{C}_{q,g}(0) \right) M$$

Trace anomaly decomposition in NNLO QCD

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)M\eta^{\mu\nu}\right]u(p)$$

$$\eta_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$2M^2 = \eta_{\mu\nu}\langle N|T^{\mu\nu}|N\rangle = \langle N|\left(\frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi\right)|N\rangle$$

$$M = \tilde{M}_q + \tilde{M}_g \qquad \tilde{M}_{q,g} = \left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)M$$

Trace anomaly decomposition in NNLO QCD

$$\eta_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0)+4\bar{C}_{q,g}(0)\right)$$

$$2M^2=\eta_{\mu\nu}\langle N|T^{\mu\nu}|N\rangle=\langle N|\left(\frac{\beta(g)}{2g}F^2+\left(1+\gamma_m(g)\right)m\bar{\psi}\psi\right)|N\rangle$$

$$M=\tilde{M}_q+\tilde{M}_g \qquad \tilde{M}_{q,g}=\left(A_{q,g}(0)+4\bar{C}_{q,g}(0)\right)M$$

	$\tilde{M}_q$	$\tilde{M}_g$	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$
proton	–0.1	1.1	0.6, 0.4	–0.18, 0.18
pion	0.4	0.6	0.6, 0.4	–0.04, 0.04

$$\mu=1.3\ \text{GeV}$$

Trace anomaly decomposition in NNLO QCD

$$\eta_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0)+4\bar{C}_{q,g}(0)\right)$$

$$2M^2 = \eta_{\mu\nu}\langle N|T^{\mu\nu}|N\rangle = \langle N|\left(\frac{\beta(g)}{2g}F^2 + (1+\gamma_m(g))m\bar{\psi}\psi\right)|N\rangle$$

$$M = \tilde{M}_q + \tilde{M}_g \qquad \tilde{M}_{q,g} = \left(A_{q,g}(0)+4\bar{C}_{q,g}(0)\right)M$$

	$\tilde{M}_q$	$\tilde{M}_g$	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$
proton	–0.1	1.1	0.6, 0.4	–0.18, 0.18
pion	0.4	0.6	0.6, 0.4	–0.04, 0.04

$\mu = 1.3$ GeV

$$\hat{H} = \hat{H}_q + \hat{H}_g = \int d^3x\,\psi^\dagger\left(-i\boldsymbol{D}\cdot\boldsymbol{\alpha}+m\beta\right)\psi + \int d^3x\,\frac{1}{2}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)$$

$$\int d^3x\,T^{00}_{q,g} = \hat{H}_{q,g}$$

$$M = M_q + M_g \qquad M_{q,g} = \left(A_{q,g}(0)+\bar{C}_{q,g}(0)\right)M$$

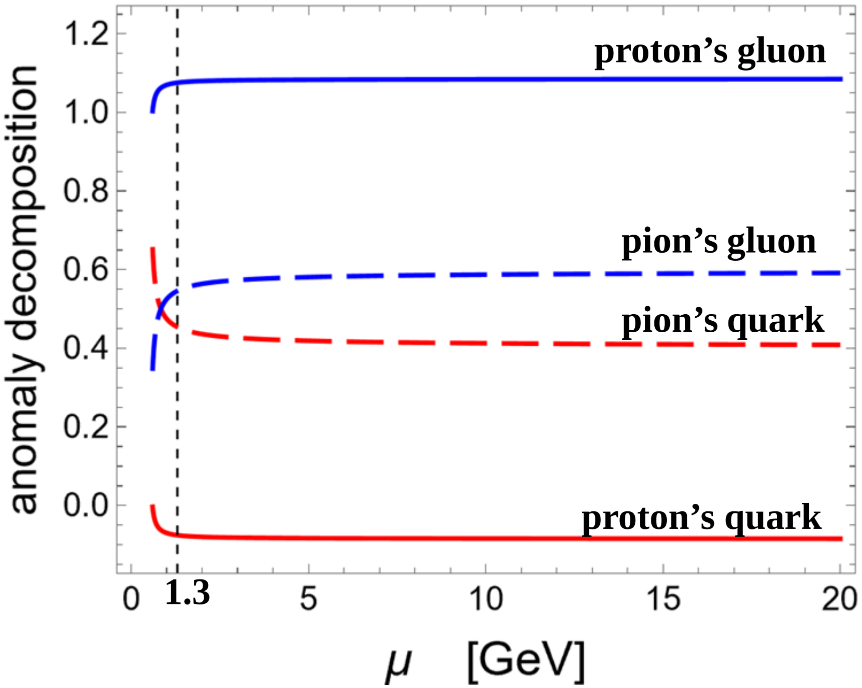
	M_q	M_g	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$
proton	0.4	0.6	0.6, 0.4	–0.18, 0.18
pion	0.6	0.4	0.6, 0.4	–0.04, 0.04

Trace anomaly decomposition in NNLO QCD

$$2M^2 = \eta_{\mu\nu}\langle N | T^{\mu\nu} | N\rangle = \langle N | \left(\frac{\beta(g)}{2g}F^2 + (1+\gamma_m(g))m\bar{\psi}\psi\right) | N\rangle$$

$$M = \tilde{M}_q + \tilde{M}_g \qquad \tilde{M}_{q,g} = \left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)M$$

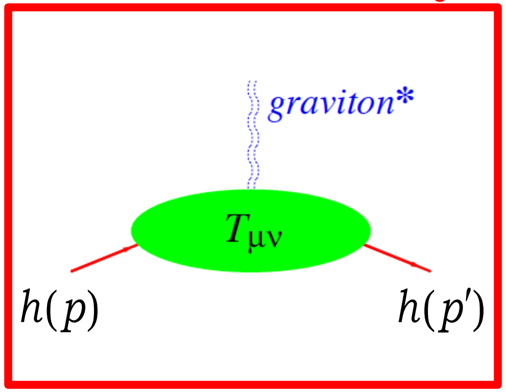
$\tilde{M}_q$	$\tilde{M}_g$	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$	
–0.1	1.1	0.6, 0.4	–0.18, 0.18	proton
0.4	0.6	0.6, 0.4	–0.04, 0.04	pion



$\overline{\text{MS}}$ scheme

Summary

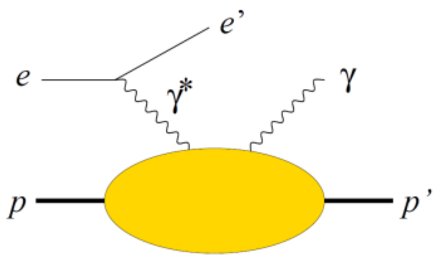
Gravitational form factors can be accessed in hard processes @ JLab, EIC, …

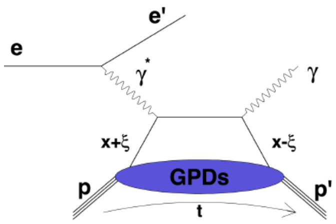


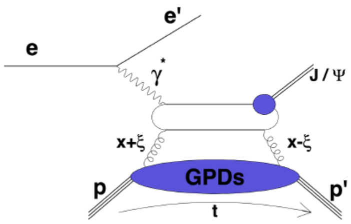
mass & energy distribution — **spin** distribution

$$\langle N(p')|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)M\eta^{\mu\nu}\Big]u(p)$$

force & pressure distribution — **mass & pressure** distribution







$$\int_{-1}^{1} dx\,xH^q(x,\eta,t) = A_q(t) + 4\eta^2 D_q(t)$$

$$\int_{-1}^{1} dx\,xE^q(x,\eta,t) = B_q(t) - 4\eta^2 D_q(t)$$

$$\bar{C}_q(0,\mu) = -\bar{C}_g(0,\mu) = \mathrm{LO} + \mathrm{NLO} + \mathrm{NNLO}$$

$$A_q(\mu_0) = \int_0^1 dx\,x[q(x,\mu_0) + \bar{q}(x,\mu_0)]$$

$$\langle N(p)|m\bar{\psi}\psi|N(p)\rangle$$

- trace anomaly for each q/g part of EMT
- NNLO term is ~1% level

$\bar{C}^N_q(0,\mu = 1\ \mathrm{GeV}) = -\bar{C}^N_g(0,\mu = 1\ \mathrm{GeV}) = -0.180 \pm 0.003$ **NNLO QCD**

$\bar{C}^\pi_q(0,\mu = 1\ \mathrm{GeV}) = -\bar{C}^\pi_g(0,\mu = 1\ \mathrm{GeV}) = -0.04 \pm 0.02$ **NNLO QCD with NLO input**

Using the NNLO values for $A_{q,g}(0), \bar{C}_{q,g}(0)$,

$T^{00} = T_q^{00} + T_g^{00}$

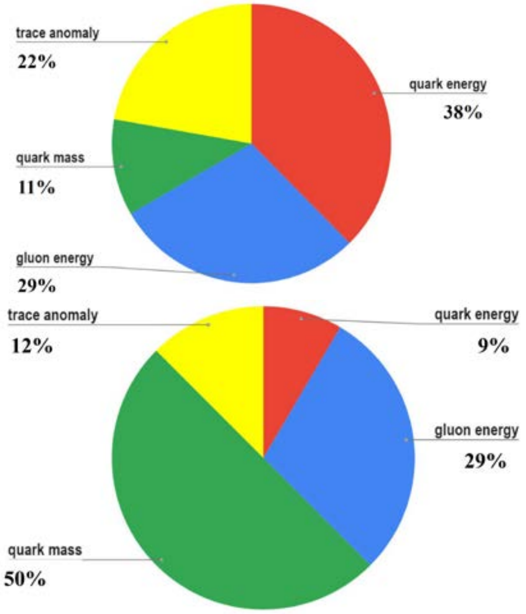
$$M = M_q + M_g \qquad M_{q,g} = \left(A_{q,g}(0) + \bar{C}_{q,g}(0)\right) M$$

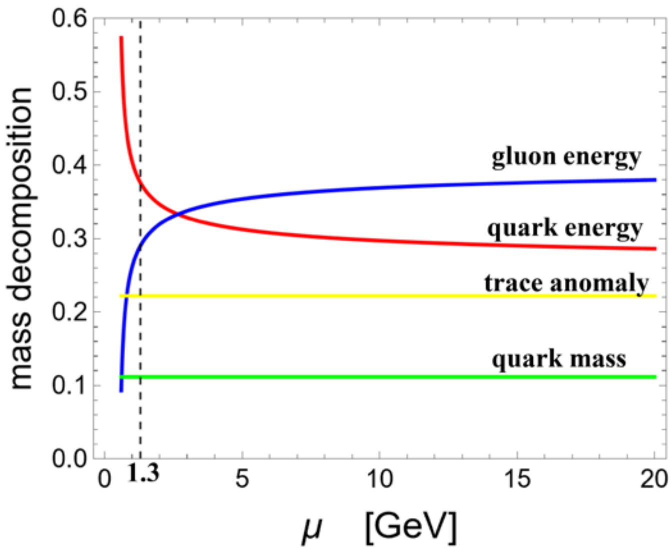
strong μ dep.

	M_q	M_g	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$
proton	0.4	0.6	0.6, 0.4	−0.18, 0.18
pion	0.6	0.4	0.6, 0.4	−0.04, 0.04

proton

pion





$\eta_{\mu\nu} T^{\mu\nu} = \eta_{\mu\nu} T_q^{\mu\nu} + \eta_{\mu\nu} T_g^{\mu\nu}$

$$M = \tilde{M}_q + \tilde{M}_g \qquad \tilde{M}_{q,g} = \left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right) M$$

weak μ dep.

	$\tilde{M}_q$	$\tilde{M}_g$	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$
proton	−0.1	1.1	0.6, 0.4	−0.18, 0.18
pion	0.4	0.6	0.6, 0.4	−0.04, 0.04

Using the NNLO values for $A_{q,g}(0), \bar{C}_{q,g}(0)$,

$T^{00} = T_q^{00} + T_g^{00}$

$$M = M_q + M_g$$

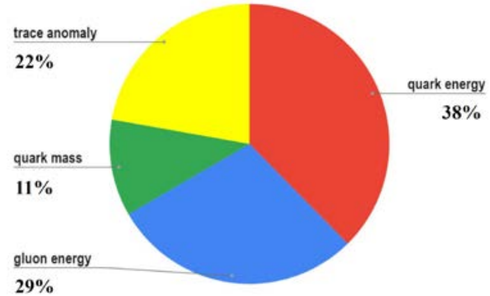
	M_q	M_g
proton	0.4	0.6
pion	0.6	0.4

$$M_{q,g} = \left(A_{q,g}(0) + \bar{C}_{q,g}(0) \right) M$$

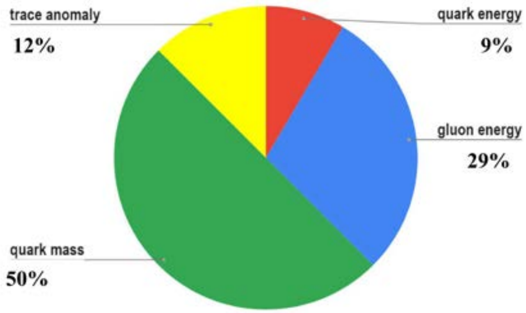
strong μ dep.

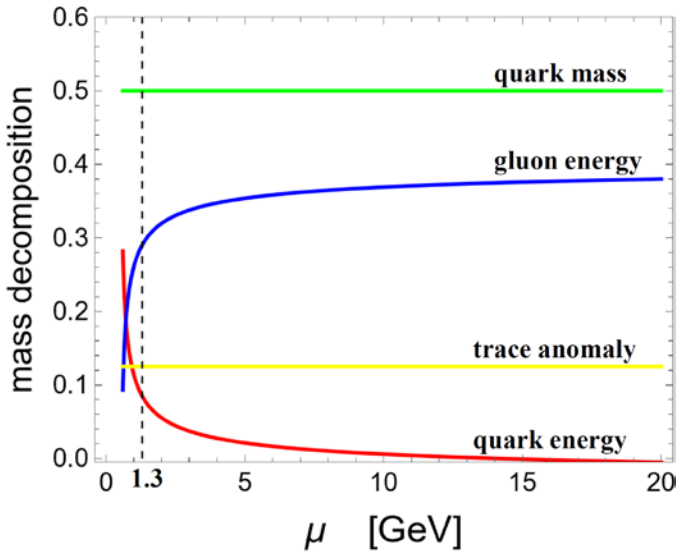
	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$
proton	0.6, 0.4	−0.18, 0.18
pion	0.6, 0.4	−0.04, 0.04

proton



pion





pion

$\eta_{\mu\nu} T^{\mu\nu} = \eta_{\mu\nu} T_q^{\mu\nu} + \eta_{\mu\nu} T_g^{\mu\nu}$

$$M = \tilde{M}_q + \tilde{M}_g$$

	$\tilde{M}_q$	$\tilde{M}_g$
proton	−0.1	1.1
pion	0.4	0.6

$$\tilde{M}_{q,g} = \left(A_{q,g}(0) + 4\bar{C}_{q,g}(0) \right) M$$

weak μ dep.

	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$
proton	0.6, 0.4	−0.18, 0.18
pion	0.6, 0.4	−0.04, 0.04

backup

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.145556 + 0.305556\frac{\langle N(p)\,|\, m\bar{\psi}\psi \,|\, N(p)\rangle}{2M^2}$$

$$+\left(0.09-0.25A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$+\alpha_s(\mu)\Big[0.00553609+0.0803962\frac{\langle N(p)\,|\, m\bar{\psi}\psi \,|\, N(p)\rangle}{2M^2}$$

$$+\left(0.0127684-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} -\left(0.0279651-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}\Big]$$

$$+\left(\alpha_s(\mu)\right)^2\Big[0.00174426+0.0312256\frac{\langle N(p)\,|\, m\bar{\psi}\psi \,|\, N(p)\rangle}{2M^2}$$

$$-\left(0.0059729-0.0165914A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$-\left(0.00396745-0.00503187A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}$$

$$+\left(0.0237481-0.0216233A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{112}{81}}\Big]$$

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.145556 + 0.305556\frac{\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle}{2M^2}$$

$$+\left(0.09-0.25A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$+\alpha_s(\mu)\Big[0.00553609 + 0.0803962\frac{\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle}{2M^2}$$

$$+\left(0.0127684-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} - \left(0.0279651-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}\Big]$$

$$+\left(\alpha_s(\mu)\right)^2\Big[0.00174426 + 0.0312256\frac{\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle}{2M^2}$$

$$-\left(0.0059729-0.0165914A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$A_q(\mu_0) = \int_0^1 dx\, x\left[q(x,\mu_0)+\bar{q}(x,\mu_0)\right]$$

$$-\left(0.00396745-0.00503187A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}$$

$$\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle = \langle N(p)|\, m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s \,|N(p)\rangle$$

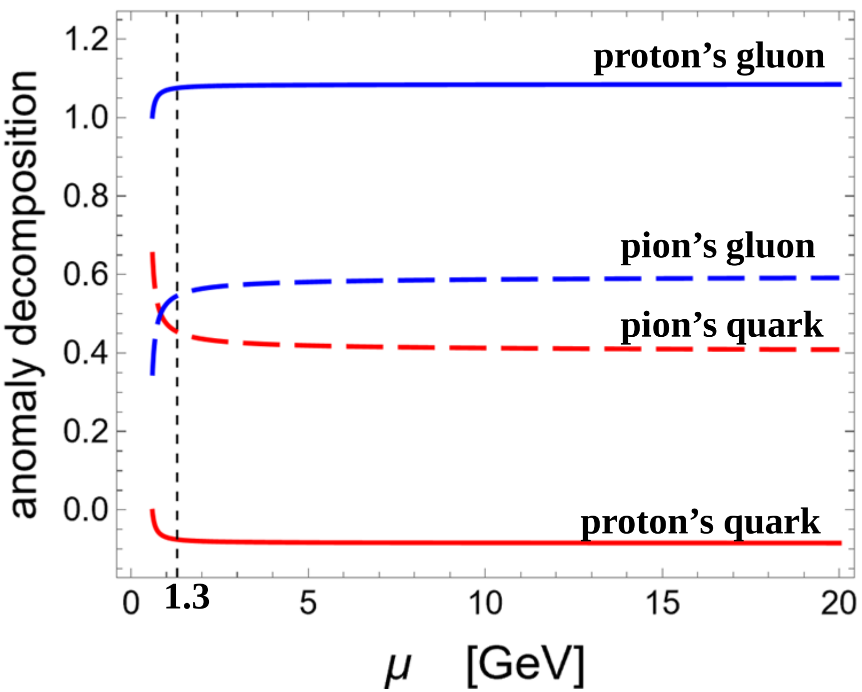
$$+\left(0.0237481-0.0216233A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{112}{81}}\Big]$$

Trace anomaly decomposition in NNLO QCD

$$2M^2 = \eta_{\mu\nu}\langle N|T^{\mu\nu}|N\rangle = \langle N|\left(\frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi\right)|N\rangle$$

$$M = \tilde{M}_q + \tilde{M}_g \qquad \tilde{M}_{q,g} = \left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)M$$

	$\tilde{M}_q$	$\tilde{M}_g$	$A_{q,g}(0)$	$\bar{C}_{q,g}(0)$
proton	-0.1	1.1	$0.6,\ 0.4$	$-0.18,\ 0.18$
pion	0.4	0.6	$0.6,\ 0.4$	$-0.04,\ 0.04$



$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{\eta^{\mu\nu}}{4}F^2$$

$$= T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$A_{q,g}(0) + 4\bar{C}_{q,g}(0) = \frac{\langle N(p)|\eta_{\mu\nu}T_{q,g}^{\mu\nu}|N(p)\rangle}{2M^2}$$

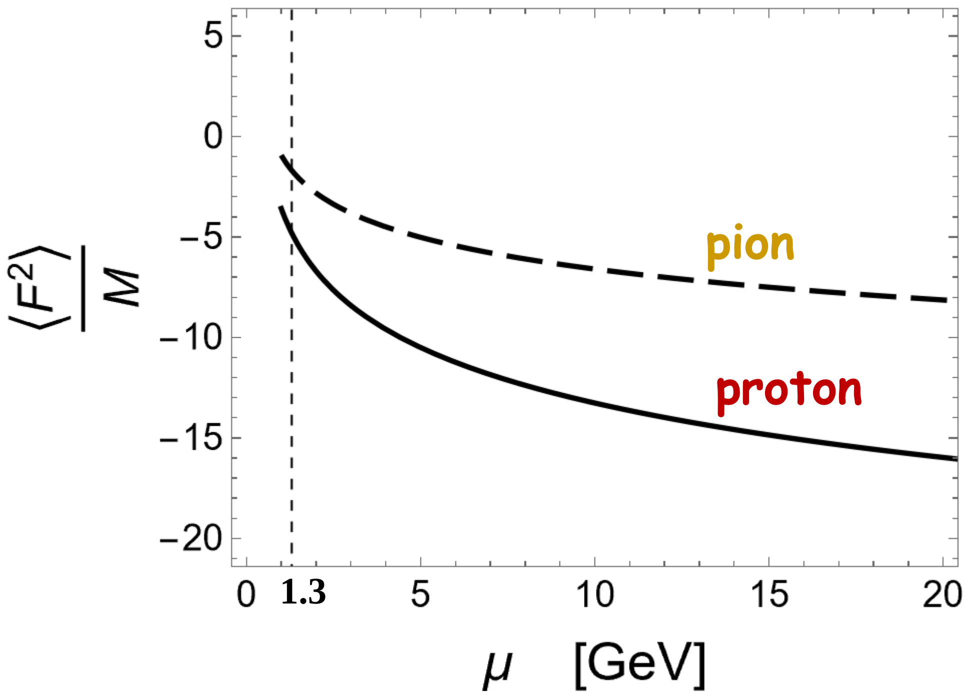
$\overline{\text{MS}}$ scheme

Hatta, Rajan, KT, JHEP 12 ('18) 008

KT, JHEP 01 ('19) 120

Gluon field in NNLO QCD

$$\frac{\langle F^2 \rangle}{M} = \frac{2g}{\beta(g)}\left(1-\left(1+\gamma_m(g)\right)\frac{\langle m\bar{\psi}\psi\rangle}{M}\right) = \begin{cases} -4.8 & \text{proton} \\ -1.7 & \text{pion} \end{cases} \quad (\mu = 1.3\ \text{GeV})$$



$\overline{\text{MS}}$ scheme

$$T^{\mu\nu}=\frac{1}{2}\overline{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi\ +\ F^{\mu\rho}F_{\rho}{}^{\nu}+\frac{\eta^{\mu\nu}}{4}F^{2}$$

$$\langle p'|T_q^{\mu\nu}|p\rangle=\frac{1}{4}\langle p'|\overline{\psi}\left(-i\overleftarrow{\partial}^{\mu}+i\overrightarrow{\partial}^{\mu}+2gA^{\mu}\right)\gamma^{\nu}\psi|p\rangle+(\mu\leftrightarrow\nu)$$

$$-i\partial^{\mu}\psi(x)=\left[\hat{P}^{\mu},\psi(x)\right]\qquad A^{\mu}(z^{-})=\frac{1}{2}\int_{-\infty}^{\infty}dz'^{-}\,\mathrm{sgn}(z'^{-}-z^{-})F^{\mu+}(z'^{-})$$

intermediated states: "partonic"

$$A_q(t)+\eta^2D_q(t)\simeq\langle x\rangle F_1^q(t)\,,\quad B_q(t)-\eta^2D_q(t)\simeq\langle x\rangle F_2^q(t)$$

$$-\frac{P^{+}\eta\Delta_{\perp}^{\mu}\overline{u}(p')u(p)}{M}D_q(t)\simeq\langle N(p')|\overline{\psi}gA_{\perp}^{\mu}\gamma^{+}\psi|N(p)\rangle$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}d\lambda\,\mathrm{sgn}(\lambda)n_{\alpha}\langle N(p')|gF_a^{\mu\alpha}(\lambda n)\overline{\psi}(0)t^{a}\gamma^{+}\psi(0)|N(p)\rangle$$

KT, PRD98, 034009

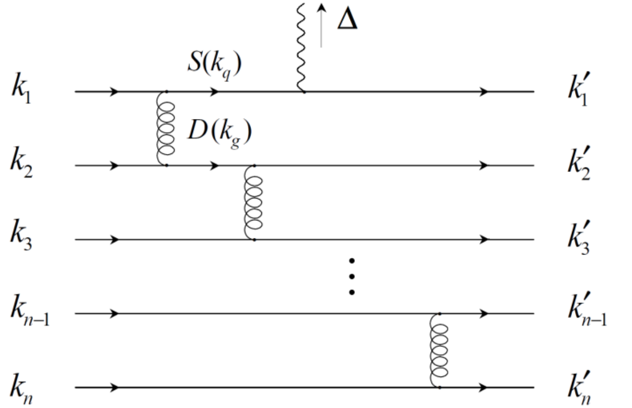
$$A_q(t)+\eta^2 D_q(t)\simeq\langle x\rangle F_1^q(t)\,,\quad B_q(t)-\eta^2 D_q(t)\simeq\langle x\rangle F_2^q(t)$$

$$-\frac{P^+\eta\Delta_\perp^\mu\bar{u}(p')u(p)}{M}D_q(t)\simeq\langle N(p')|\bar{\psi}gA_\perp^\mu\gamma^+ q\psi|N(p)\rangle$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}d\lambda\,\mathrm{sgn}(\lambda)n_\alpha\langle N(p')|gF_a^{\mu\alpha}(\lambda n)\bar{\psi}(0)t^a\gamma^+\psi(0)|N(p)\rangle$$

$t\to\infty$

$$A_q(t)\sim\frac{1}{t^2}\,,\qquad D_q(t)\sim\frac{1}{t^3}$$



$$2M^2 = \langle N \,|\left(\frac{\beta(g)}{2g}F^2 + \big(1+\gamma_m(g)\big)m\bar{\psi}\psi\right)|\, N\rangle \simeq \langle N\,|\,\frac{\beta(g)}{2g}F^2\,|\,N\rangle$$

$$2M^2 = \eta_{\mu\nu}\langle N\,|\,T_q^{\mu\nu}\,|\,N\rangle + \eta_{\mu\nu}\langle N\,|\,T_g^{\mu\nu}\,|\,N\rangle$$

KT, JHEP1901, 120

$$2M^2 = \langle N | \left(\frac{\beta(g)}{2g} F^2 + \big(1+\gamma_m(g)\big) m\bar{\psi}\psi \right) | N \rangle \simeq \langle N | \frac{\beta(g)}{2g} F^2 | N \rangle$$

$$2M^2 = \eta_{\mu\nu} \langle N | T_q^{\mu\nu} | N \rangle + \eta_{\mu\nu} \langle N | T_g^{\mu\nu} | N \rangle$$

1-loop

$$\frac{\alpha_s}{4\pi} \frac{n_f}{3} F^2 \qquad \frac{\alpha_s}{4\pi} \left(-\frac{11 C_A}{6} F^2 \right)$$

KT, JHEP1901, 120

$$2M^2 = \langle N | \left(\frac{\beta(g)}{2g} F^2 + \left(1+\gamma_m(g)\right) m\bar{\psi}\psi \right) | N \rangle \simeq \langle N | \frac{\beta(g)}{2g} F^2 | N \rangle$$

$$2M^2 = \eta_{\mu\nu} \langle N | T_q^{\mu\nu} | N \rangle + \eta_{\mu\nu} \langle N | T_g^{\mu\nu} | N \rangle$$

1-loop

$$\frac{\alpha_s}{4\pi} \frac{n_f}{3} F^2 \qquad \frac{\alpha_s}{4\pi}\left(-\frac{11C_A}{6} F^2 \right)$$

2-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54} \right) F^2 \qquad \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54} \right) F^2$$

3-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^3 \left[\left\{ n_f \left(\frac{52\zeta(3)}{9} - \frac{401}{324} \right) C_A C_F + \left(\frac{134}{27} - 4\zeta(3) \right) C_A^2 + \left(\frac{2407}{1458} - \frac{16\zeta(3)}{9} \right) C_F^2 \right\} + n_f^2 \left(-\frac{697C_A}{729} - \frac{169C_F}{1458} \right) \right] F^2$$

$$\left(\frac{\alpha_s}{4\pi}\right)^3 \left[n_f \left(\left(\frac{1123}{162} - \frac{52\zeta(3)}{9} \right) C_A C_F + \left(4\zeta(3) + \frac{293}{36} \right) C_A^2 + \frac{16}{729}(81\zeta(3) - 98) C_F^2 \right) + n_f^2 \left(\frac{655C_A}{2916} - \frac{361C_F}{729} \right) - \frac{2857C_A^3}{108} \right] F^2$$

KT, JHEP1901, 120

$$2M^2 = \langle N | \left(\frac{\beta(g)}{2g} F^2 + \left(1+\gamma_m(g)\right) m\bar{\psi}\psi \right) | N \rangle \simeq \langle N | \frac{\beta(g)}{2g} F^2 | N \rangle$$

$$2M^2 = \eta_{\mu\nu} \langle N | T_q^{\mu\nu} | N \rangle + \eta_{\mu\nu} \langle N | T_g^{\mu\nu} | N \rangle$$

1-loop

$$\frac{\alpha_s}{4\pi} \frac{n_f}{3} F^2 \qquad \frac{\alpha_s}{4\pi} \left(-\frac{11 C_A}{6} F^2 \right)$$

2-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{17 C_A n_f}{27} + \frac{49 C_F n_f}{54} \right) F^2 \qquad \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{28 C_A n_f}{27} - \frac{17 C_A^2}{3} + \frac{5 C_F n_f}{54} \right) F^2$$

3-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^3 \left[\left\{ n_f \left(\frac{52\zeta(3)}{9} - \frac{401}{324} \right) C_A C_F + \left(\frac{134}{27} - 4\zeta(3) \right) C_A^2 + \left(\frac{2407}{1458} - \frac{16\zeta(3)}{9} \right) C_F^2 \right\} + n_f^2 \left(-\frac{697 C_A}{729} - \frac{169 C_F}{1458} \right) \right] F^2$$

$$\left(\frac{\alpha_s}{4\pi}\right)^3 \left[n_f \left(\left(\frac{1123}{162} - \frac{52\zeta(3)}{9} \right) C_A C_F + \left(4\zeta(3) + \frac{293}{36} \right) C_A^2 + \frac{16}{729} (81\zeta(3) - 98) C_F^2 \right) + n_f^2 \left(\frac{655 C_A}{2916} - \frac{361 C_F}{729} \right) - \frac{2857 C_A^3}{108} \right] F^2$$

nucleon -1 : 5

KT, JHEP1901, 120

$$2M^2 = \langle N | \left(\frac{\beta(g)}{2g} F^2 + (1+\gamma_m(g)) m\bar{\psi}\psi \right) | N\rangle \simeq \langle N | \frac{\beta(g)}{2g} F^2 | N\rangle$$

$$2M^2 = \eta_{\mu\nu}\langle N | T_q^{\mu\nu} | N\rangle + \eta_{\mu\nu}\langle N | T_g^{\mu\nu} | N\rangle$$

1-loop

$$\frac{\alpha_s}{4\pi}\frac{n_f}{3}F^2 \qquad \frac{\alpha_s}{4\pi}\left(-\frac{11C_A}{6}F^2\right)$$

2-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{17C_A n_f}{27}+\frac{49C_F n_f}{54}\right)F^2 \qquad \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{28C_A n_f}{27}-\frac{17C_A^2}{3}+\frac{5C_F n_f}{54}\right)F^2$$

3-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\frac{52\zeta(3)}{9}-\frac{401}{324}\right)C_A C_F + \left(\frac{134}{27}-4\zeta(3)\right)C_A^2 + \left(\frac{2407}{1458}-\frac{16\zeta(3)}{9}\right)C_F^2\right\} + n_f^2\left(-\frac{697C_A}{729}-\frac{169C_F}{1458}\right)\right]F^2$$

$$\left(\frac{\alpha_s}{4\pi}\right)^3\left[n_f\left(\left(\frac{1123}{162}-\frac{52\zeta(3)}{9}\right)C_A C_F + \left(4\zeta(3)+\frac{293}{36}\right)C_A^2 + \frac{16}{729}(81\zeta(3)-98)C_F^2\right) + n_f^2\left(\frac{655C_A}{2916}-\frac{361C_F}{729}\right) - \frac{2857C_A^3}{108}\right]F^2$$

nucleon -1 : 5

$$2m_\pi^2 = \eta_{\mu\nu}\langle \pi | T_q^{\mu\nu} | \pi\rangle + \eta_{\mu\nu}\langle \pi | T_g^{\mu\nu} | \pi\rangle$$

KT, JHEP1901, 120

$$2M^2 = \langle N | \left(\frac{\beta(g)}{2g} F^2 + (1+\gamma_m(g)) m\bar{\psi}\psi \right) | N \rangle \simeq \langle N | \frac{\beta(g)}{2g} F^2 | N \rangle$$

$$2M^2 = \eta_{\mu\nu} \langle N | T_q^{\mu\nu} | N \rangle + \eta_{\mu\nu} \langle N | T_g^{\mu\nu} | N \rangle$$

1-loop

$$\frac{\alpha_s}{4\pi} \frac{n_f}{3} F^2 \qquad \frac{\alpha_s}{4\pi}\left(-\frac{11C_A}{6} F^2 \right)$$

2-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54} \right) F^2 \qquad \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54} \right) F^2$$

3-loop

$$\left(\frac{\alpha_s}{4\pi}\right)^3 \left[\left\{ n_f \left(\frac{52\zeta(3)}{9} - \frac{401}{324} \right) C_A C_F + \left(\frac{134}{27} - 4\zeta(3) \right) C_A^2 + \left(\frac{2407}{1458} - \frac{16\zeta(3)}{9} \right) C_F^2 \right\} + n_f^2 \left(-\frac{697C_A}{729} - \frac{169C_F}{1458} \right) \right] F^2$$

$$\left(\frac{\alpha_s}{4\pi}\right)^3 \left[n_f \left(\left(\frac{1123}{162} - \frac{52\zeta(3)}{9} \right) C_A C_F + \left(4\zeta(3) + \frac{293}{36} \right) C_A^2 + \frac{16}{729}(81\zeta(3) - 98) C_F^2 \right) + n_f^2 \left(\frac{655C_A}{2916} - \frac{361C_F}{729} \right) - \frac{2857C_A^3}{108} \right] F^2$$

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$$2m_\pi^2 = \eta_{\mu\nu} \langle \pi | T_q^{\mu\nu} | \pi \rangle + \eta_{\mu\nu} \langle \pi | T_g^{\mu\nu} | \pi \rangle$$

KT, JHEP1901, 120