Attractor Behavior and Thermodynamic Stability in Spin Hydrodynamics

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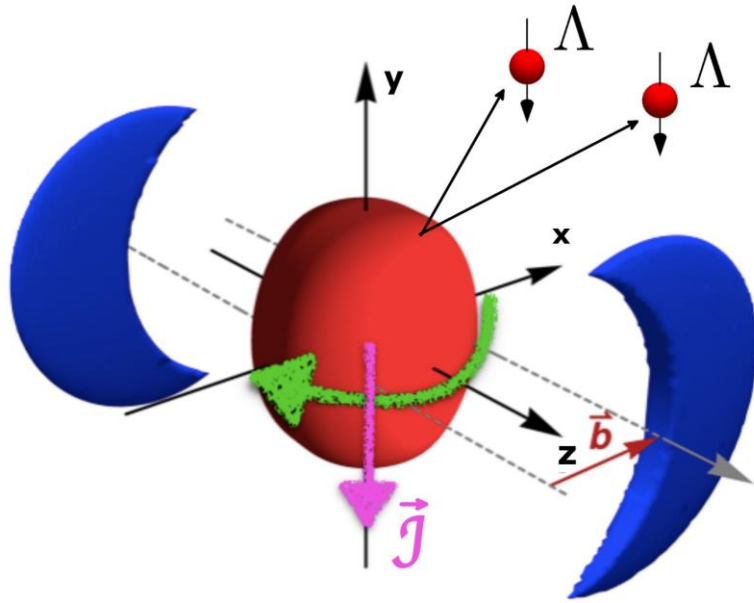**The 12th Circum-Pan-Pacific Symposium on High Energy Spin Physics**
November 8-12, 2024

Outline

- ➢ **Introduction**

- ➢ **Evolution of spin density in spin hydrodynamics**

- ➢ **Causality and stability analysis for spin Hydrodynamics**

- ➢ **Summary**

Introduction

Rotation and Polarization

picture from Florkowski, Kumar, Ryblewski, PPNP (2019)

➢ **Large angular momentum in noncentral heavy ion collisions**

$$J \sim \frac{A\sqrt{s}}{2} b \sim 10^5 \hbar$$
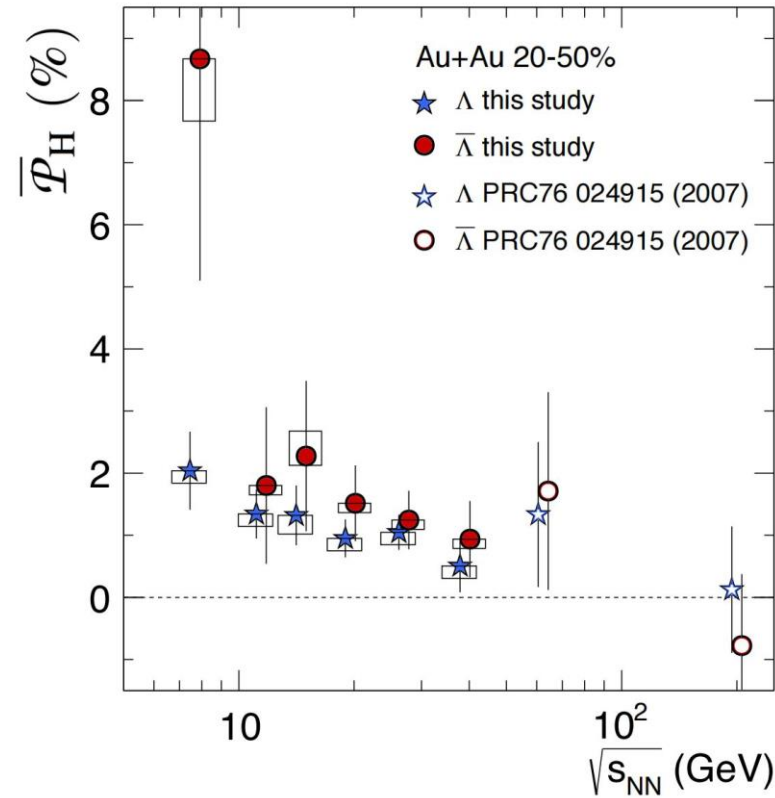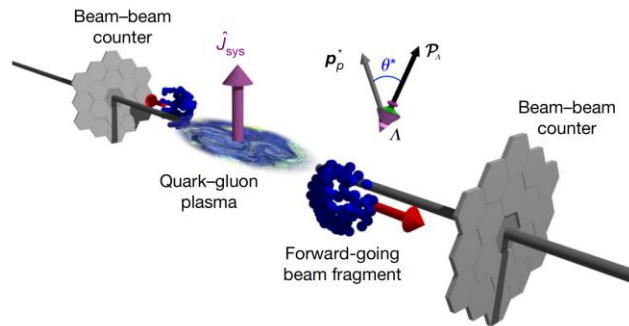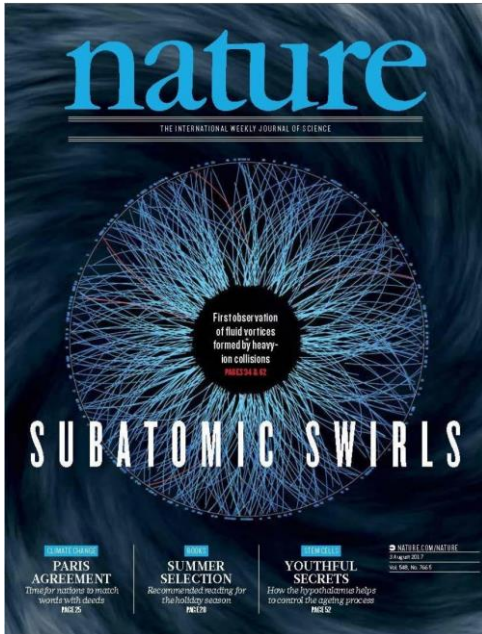
Vorticity of quark gluon plasma

⇩ (Total angular momentum conservation)

Polarization of particles along vorticity

Liang, Wang, PRL (2005); PLB (2005)
Gao, Chen, Deng, Liang, Wang, Wang, PRC (2008)

Most Vortical Fluid and Spin Polarization

> ➤ **Weak decay:** $\Lambda \to p + \pi^-$

> ➤ **Proton distribution:**

$$\frac{dN}{d\cos\theta^*} = \frac{1}{2}(1 + \alpha_H |\vec{\mathcal{P}}_H| \cos\theta^*)$$
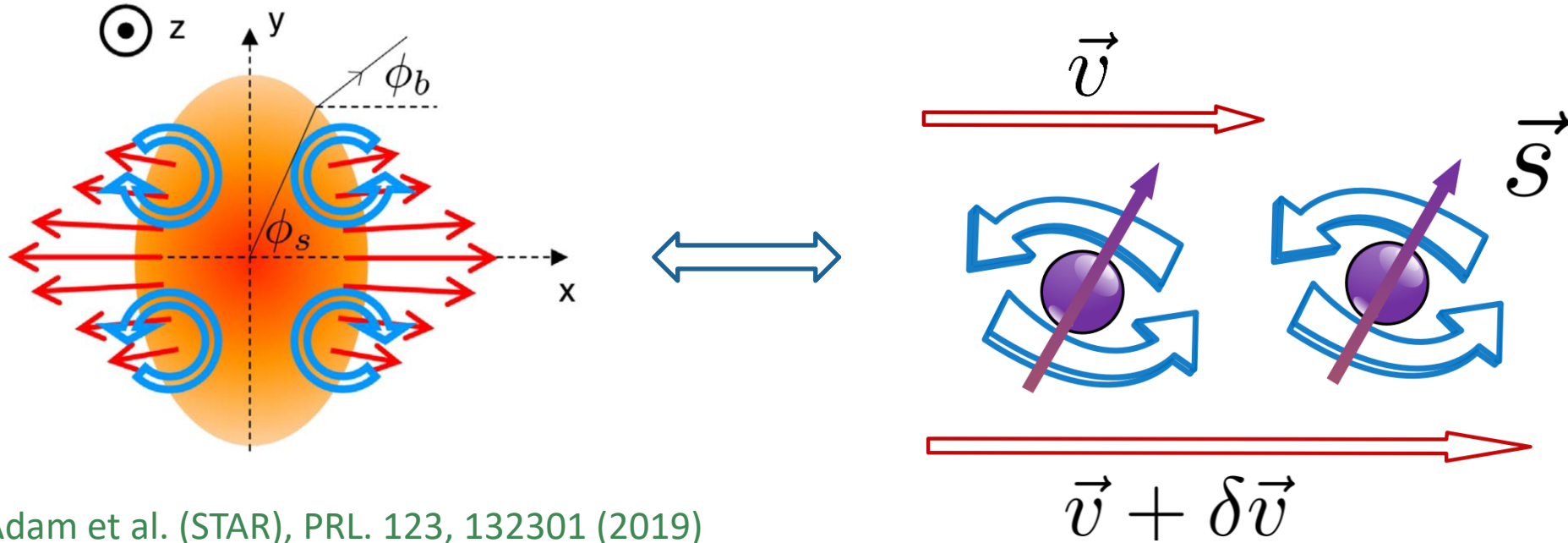
> ➤ **Large vorticity:**

$$\omega \approx (9 \pm 1) \times 10^{21} \text{s}^{-1}$$

Most vortical fluid!

Adamczyk et al. (STAR), Nature 548, 62 (2017)

Spin-Orbit Coupling in Hydrodynamics

J. Adam et al. (STAR), PRL. 123, 132301 (2019)

How to study the transform between spin and orbital angular momentum at hydrodynamic level?

Relativistic Spin Hydrodynamics

Modified thermodynamic relations:

$$e + p = Ts + \mu n + \underline{\omega_{\mu\nu} S^{\mu\nu}}$$
$$de = Tds + \mu dn + \underline{\omega_{\mu\nu} dS^{\mu\nu}}$$

energy density: e pressure: p

temperature: T entropy density: s

spin density: $S^{\mu\nu}$ spin chemical potential: $\omega_{\mu\nu}$

Florkowski, Kumar, Ryblewsk, PPNP (2019)

Hattori, Hongo, Huang, Matsuo, Taya, PLB (2019)

Hongo, Huang, Kaminski, Stephanov, Yee, JHEP (2021)

Fukushima, Pu, PLB (2021)

Conservation equations:

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0$$

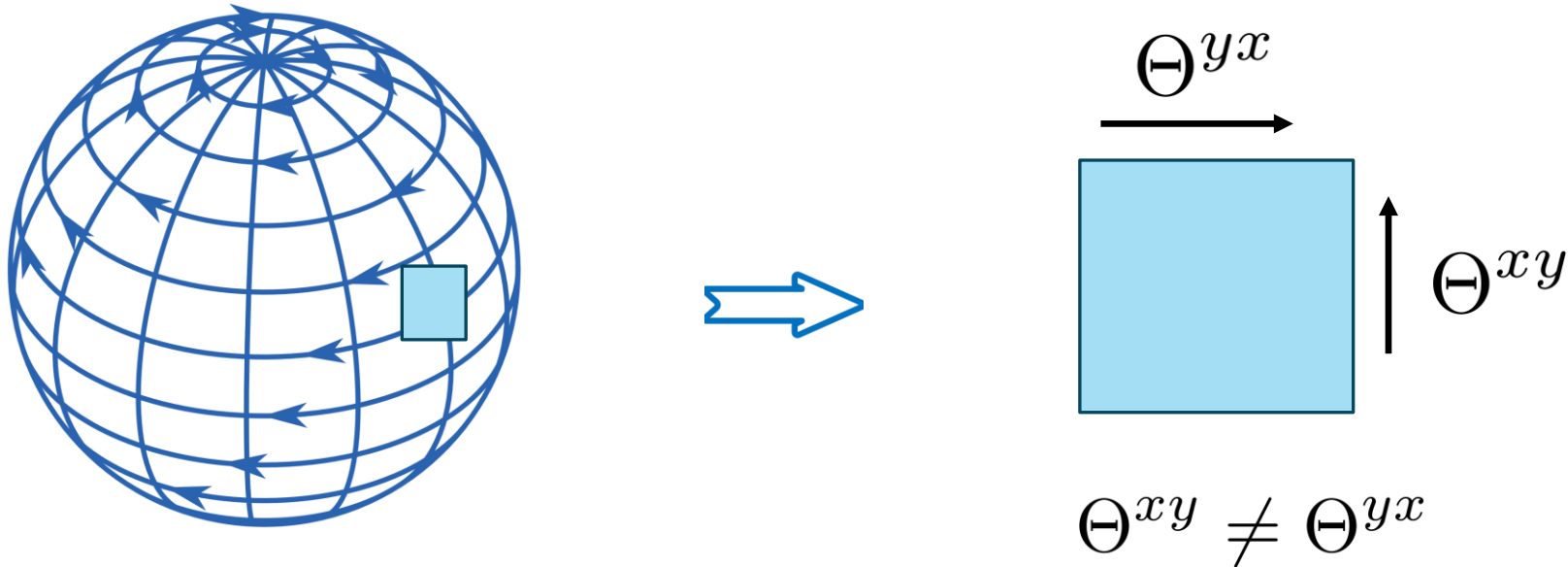energy-momentum   particle number   total angular momentum

$$J^{\lambda\mu\nu} = x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu} + \Sigma^{\lambda\mu\nu}$$

$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma_\perp^{\lambda\mu\nu}$$

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$$

Convertibility between spin and orbital angular momentum

Antisymmetric Part of EMT

$$\Theta^{yx}$$

$$\Theta^{xy}$$

$$\Theta^{xy} \neq \Theta^{yx}$$

There exists nonzero net torque acting on the fluid cell

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$$

Problems in Spin Hydrodynamics

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0$$

$$h^\mu - \frac{e+p}{n}\nu^\mu = \kappa[\Delta^{\mu\nu}\partial_\nu T - T(u\cdot\partial)u^\mu],$$

$$\pi^{\mu\nu} = \eta_s \partial^{<\mu} u^{\nu>} + \zeta(\partial\cdot u)\Delta^{\mu\nu},$$

$$q^\mu = \lambda\left[\frac{1}{T}\Delta^{\mu\nu}\partial_\nu T + (u\cdot\partial)u^\mu - 4\omega^{\mu\nu}u_\nu\right],$$

$$\phi^{\mu\nu} = -\gamma\left(\Omega^{\mu\nu} - \frac{2}{T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta}\right),$$

Florkowski, Ryblewsk, Kumar, PPNP (2019)
Hattori, Hongo, Huang, Matsuo, Taya, PLB (2019)
Hongo, Huang, Kaminski, Stephanov, Yee, JHEP (2021)
Fukushima, Pu, PLB (2021)

1. **Can spin density be considered a hydrodynamic variable?**

2. **Initial conditions for spin hydrodynamics are still unknown.**

3. **How to ensure causality and stability in spin hydrodynamics?**

4. **....**

Systematic studies for spin hydrodynamics are required!

Evolution of Spin Density in Spin Hydrodynamics

➢ **Regime for spin hydrodynamics**

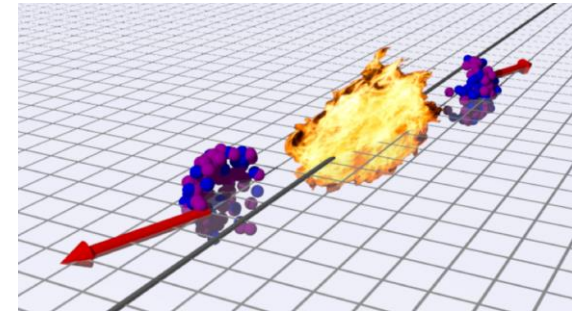➢ **Late-time attractor**

Spin Hydrodynamics in Bjorken Flow

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0$$

$$e = 3p$$
$$S^{\mu\nu} = aT^2 \omega^{\mu\nu}$$
$$n \sim 0$$

$$u^\mu = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

picture from Becattini, Lisa, ARNPS (2020)

J. D. Bjorken, Phys. Rev. D 27, 140(1983)

z

➤ **Symmetric constraints expressed in Lie derivative**

$$\mathcal{L}_\kappa S^{\mu\nu} = 0 \qquad \kappa \in \{\partial_x, \ \partial_y, \ x\partial_y - y\partial_x, \ z\partial_t + t\partial_z\}$$

➤ **The hydrodynamic equations in a Bjorken flow are self-consistent iff**

$$S^{\mu\nu} = 0 \text{ for } (\mu\nu) \neq (xy)$$

DLW, Fang, Pu, PRD 104, 11404 (2021)
DLW, Yan, Pu, arXiv: 2408.03781

Asymptotic Analysis for Spin Hydrodynamics

DLW, Yan, Pu, arXiv: 2408.03781

➢ **Parameterize time-dependent coefficients:** $\mathrm{Kn}^{-1} \approx w \equiv \dfrac{\tau}{\tau_\phi} = \left(\dfrac{\tau}{\tau_1}\right)^{\Delta_1}, \quad \dfrac{\tau_\phi \gamma}{\chi} = \alpha w^{\Delta_2}$

➢ **Late-time asymptotic solutions for $\Delta_1 > 0$:**

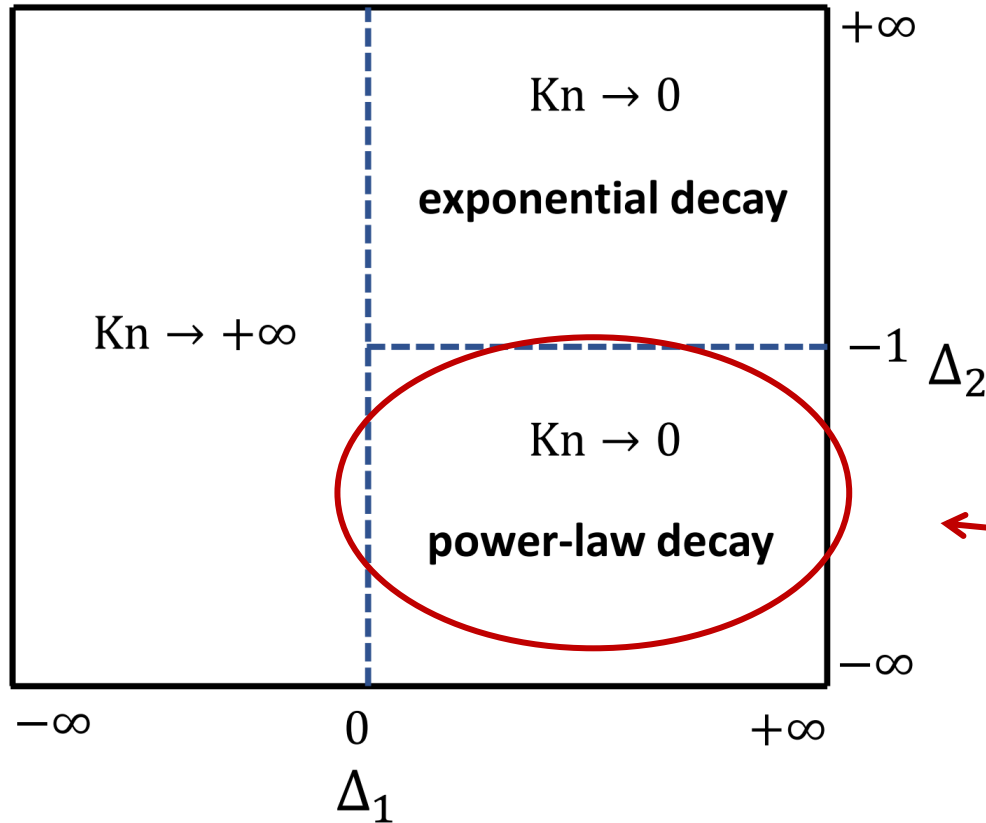| $\Delta_2 = -1$	$S_{(1)} \propto e^{-w/\Delta_1}, \; S_{(2)} \sim w^{-(1+8\alpha)/\Delta_1}$
$\Delta_2 < -1$	$S_{(1)} \propto e^{-w/\Delta_1}, \; S_{(2)} \sim w^{-1/\Delta_1}$

$\Delta_2 = 0$	$S_{(1),(2)} \propto e^{-w(1\pm\sqrt{1-32\alpha})/(2\Delta_1)}$
$-1 < \Delta_2 < 0$	$S_{(1)} \propto e^{-w/\Delta_1}, \; S_{(2)} \propto \exp\left[-\dfrac{8\alpha w^{1+\Delta_2}}{\Delta_1(1+\Delta_2)}\right]$
$\Delta_2 = -1$	$S_{(1)} \propto e^{-w/\Delta_1}, \; S_{(2)} \sim w^{-(1+8\alpha)/\Delta_1}$

power-law decay (hydro-mode)

exponential decay (non-hydro-mode)

Regime for Spin Hydrodynamics

DLW, Yan, Pu, arXiv: 2408.03781

$$\mathrm{Kn}^{-1} \approx w \equiv \frac{\tau}{\tau_\phi} = \left(\frac{\tau}{\tau_1}\right)^{\Delta_1}, \quad \frac{\tau_\phi \gamma}{\chi} = \alpha w^{\Delta_2}$$

Spin hydrodynamics is well-defined in this regime!
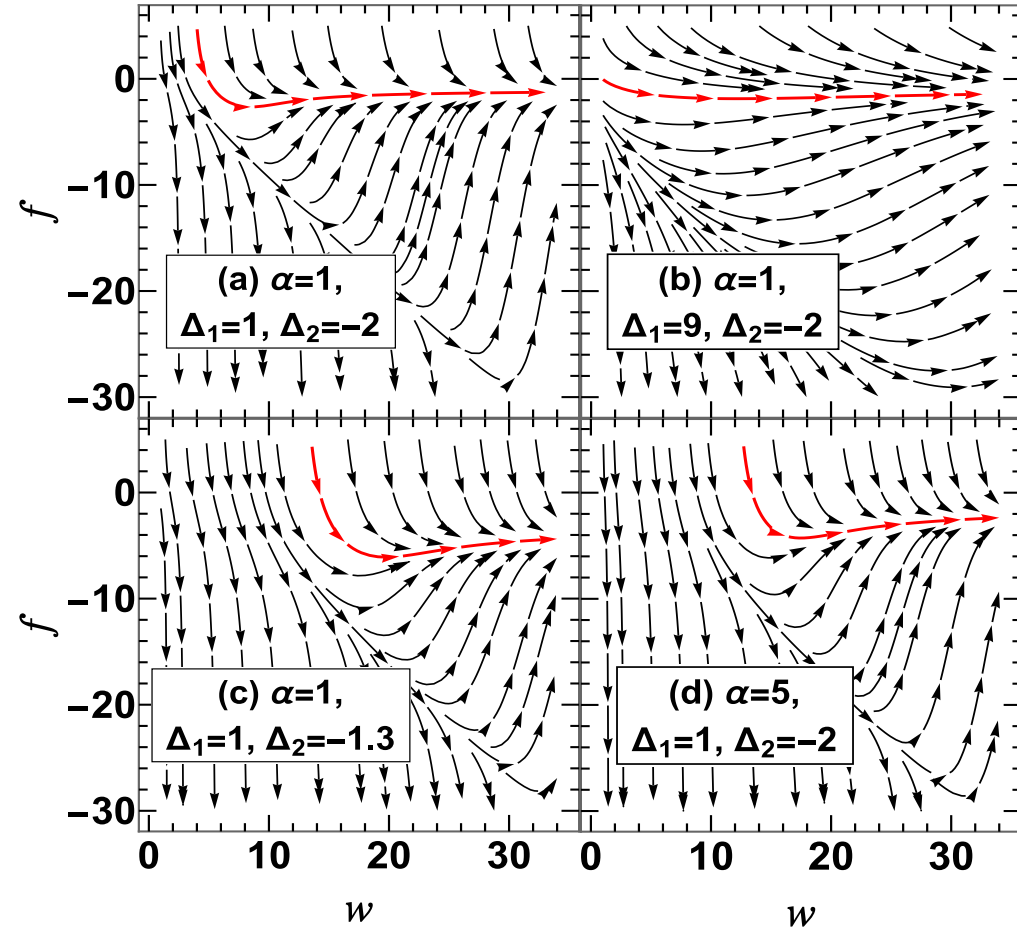
Late-Time Attractor Behavior

➢ **Decay rate of spin density:**

$$f(w) \equiv \Delta_1 \frac{w}{S^{xy}} \frac{dS^{xy}}{dw} = \frac{\tau}{S^{xy}} \frac{dS^{xy}}{d\tau}$$

➢ **Attractor solution:**

$$f \to \begin{cases} -1 - 8\alpha, & \Delta_2 = -1, \\ -1, & \Delta_2 < -1. \end{cases}$$

Late-time decay rate is
insensitive to initial conditions!

(a) $\alpha=1$, $\Delta_1=1$, $\Delta_2=-2$

(b) $\alpha=1$, $\Delta_1=9$, $\Delta_2=-2$

(c) $\alpha=1$, $\Delta_1=1$, $\Delta_2=-1.3$

(d) $\alpha=5$, $\Delta_1=1$, $\Delta_2=-2$
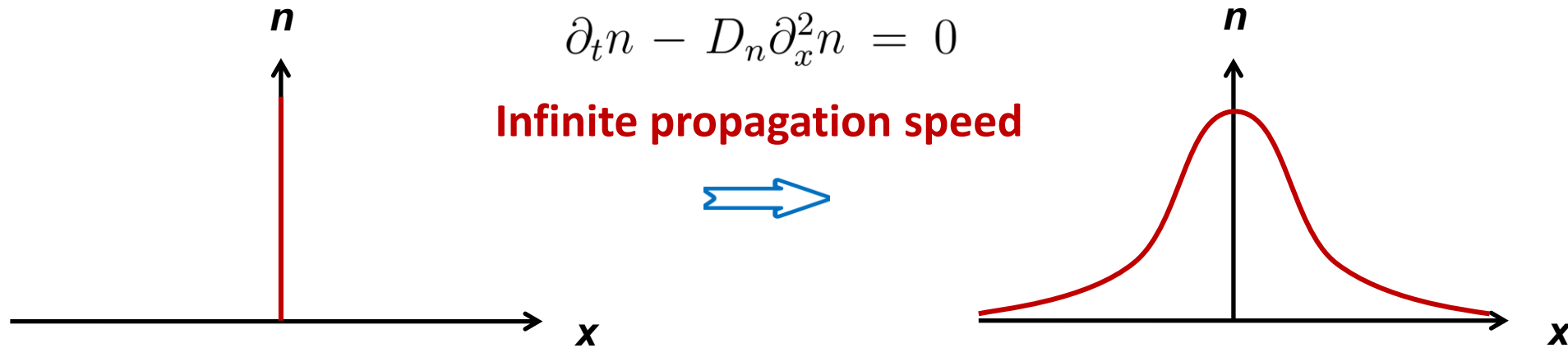
DLW, Yan, Pu, arXiv: 2408.03781

Causality and Stability Analysis for Spin Hydrodynamics

➢ **Acausal modes in first order spin hydrodynamics**

➢ **Remove acausal modes**

➢ **Thermodynamic stability analysis**

Example: Non-relativistic Diffusion

$$n(t = 0, x) = \delta(x)$$

$$n(t > 0, x) = \frac{1}{2\sqrt{\pi D_n t}} \exp\left(-\frac{x^2}{4 D_n t}\right)$$

$$\partial_t n - D_n \partial_x^2 n = 0$$

Infinite propagation speed

n

x

n

x

➢ **The initial data $n(0, x)$ is zero for $x \neq 0$**

➢ **For any small time $t > 0$, $n(t, x)$ is nonzero everywhere**

Acausal Modes in First Order Spin Hydrodynamics

➢ **Independent small perturbations on top of equilibrium satisfy**

$$\partial_\mu \delta\Theta^{\mu\nu} = 0, \quad \partial_\lambda \delta J^{\lambda\mu\nu} = 0, \quad \partial_\mu \delta j^\mu = 0.$$

➢ **One mode behaves like non-relativistic diffusion**

$$\omega \propto k^2$$

Infinite propagation speed!

$$\partial_t n - D_n \partial_x^2 n = 0$$
$$\Downarrow$$
$$\omega = -i D_n k^2$$

Krotscheck, Kundt, Communications in Mathematical Physics 60, 171 (1978)
Xie, DLW, Yang, Pu, PRD 108, 094031 (2023); DLW, Pu, PRD 109, L031504 (2024)

Remove Modes with Infinite Propagation Speed

➢ **Introduce nonzero relaxation times:**

$$\tau_q \Delta^{\mu\nu} \frac{d}{d\tau} q_\nu + q^\mu = \lambda(T^{-1}\Delta^{\mu\alpha}\partial_\alpha T + Du^\mu - 4\omega^{\mu\nu}u_\nu),$$

$$\tau_\phi \Delta^{\mu\alpha}\Delta^{\nu\beta} \frac{d}{d\tau} \phi_{\alpha\beta} + \phi^{\mu\nu} = 2\gamma_s \Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_{[\alpha}u_{\beta]} + 2\omega_{\alpha\beta}),$$

$$\tau_\pi \Delta^{\alpha<\mu}\Delta^{\nu>\beta} \frac{d}{d\tau} \pi_{\alpha\beta} + \pi^{\mu\nu} = 2\eta\partial^{<\mu}u^{\nu>},$$

$$\tau_\Pi \frac{d}{d\tau}\Pi + \Pi = -\zeta\partial_\mu u^\mu,$$

Liu, Huang, Nucl. Sci. Tech. (2020)
Xie, DLW, Yang, Pu, PRD 108, 094031 (2023)

➢ **The thermodynamic flux cannot instantaneously vanish/appear when thermodynamic force is suddenly switched off/on.**

Israel, Stewart, Annals Phys. (1979); Muronga, PRC (2004); Koide, Denicol, Mota, Kodama, PRC (2007)
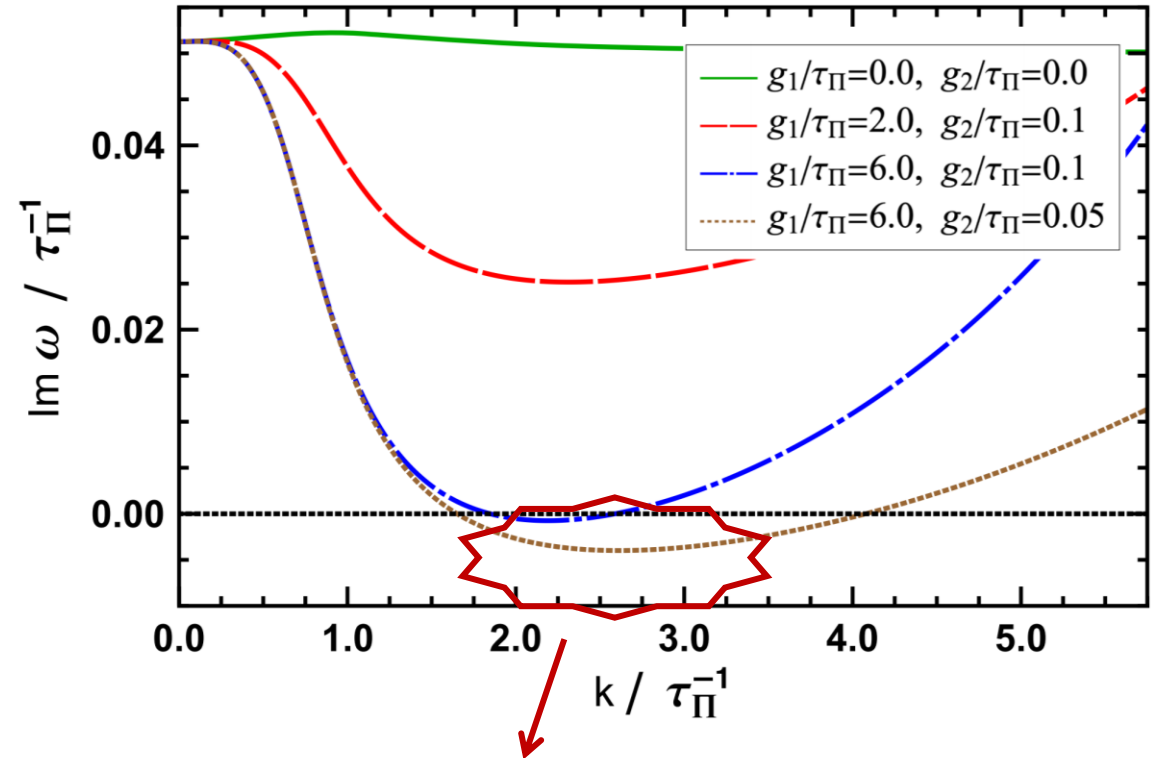
Is Equilibrium State Unstable?

➢ **Check stability by linear mode analysis**

$$\delta\varphi \sim e^{i\omega t - ikx}, \quad \text{Im}\, \omega(k) > 0$$

➢ **Complicated dispersion relations.**

➢ **Necessary conditions are obtainable, but they are insufficient for stability.**

Xie, DLW, Yang, Pu, PRD 108, 094031 (2023)
Daher, Florkowski, Ryblewski, Taghinavaz, PRD 109, 114001 (2024)

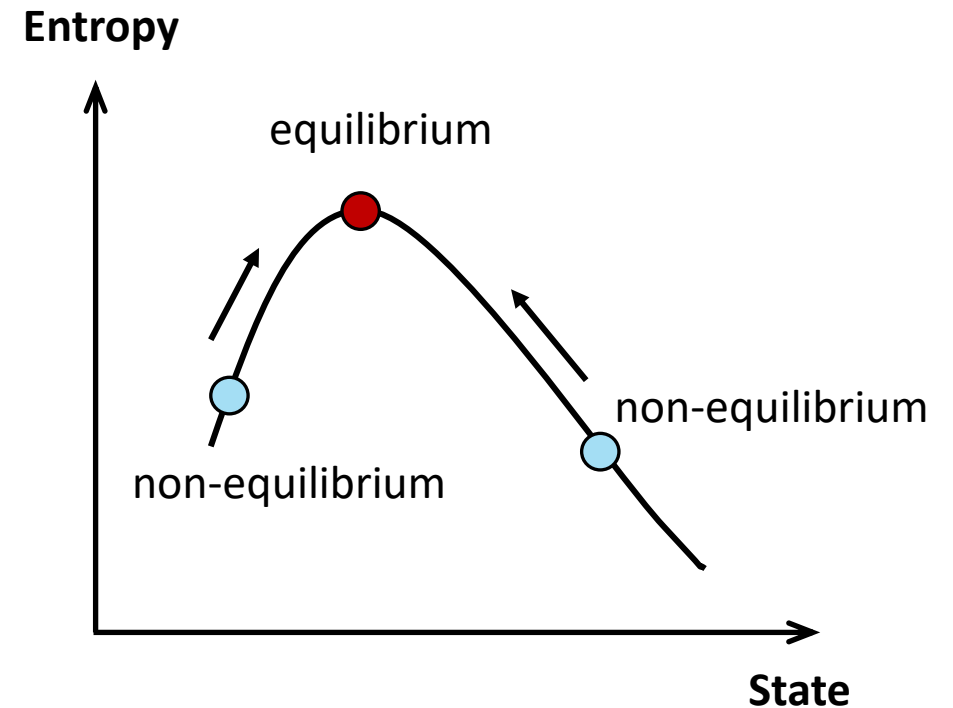**Unstable!**

Method of Thermodynamic Stability

➤ **Entropy principle:**

$$\Delta S_{\text{bath}} = \alpha_I \Delta Q^I_{\text{fluid}}$$

$$\downarrow$$

$$\Delta S_{\text{fluid}} + \Delta S_{\text{bath}} \geq 0$$

$$\Psi = S_{\text{fluid}} + \alpha_I Q^I_{\text{fluid}}$$

Ψ {
  maximized in equilibrium state

  non-decreasing functional
}

Gavassino, Antonelli, Haskell, PRL 128, 010606 (2022)
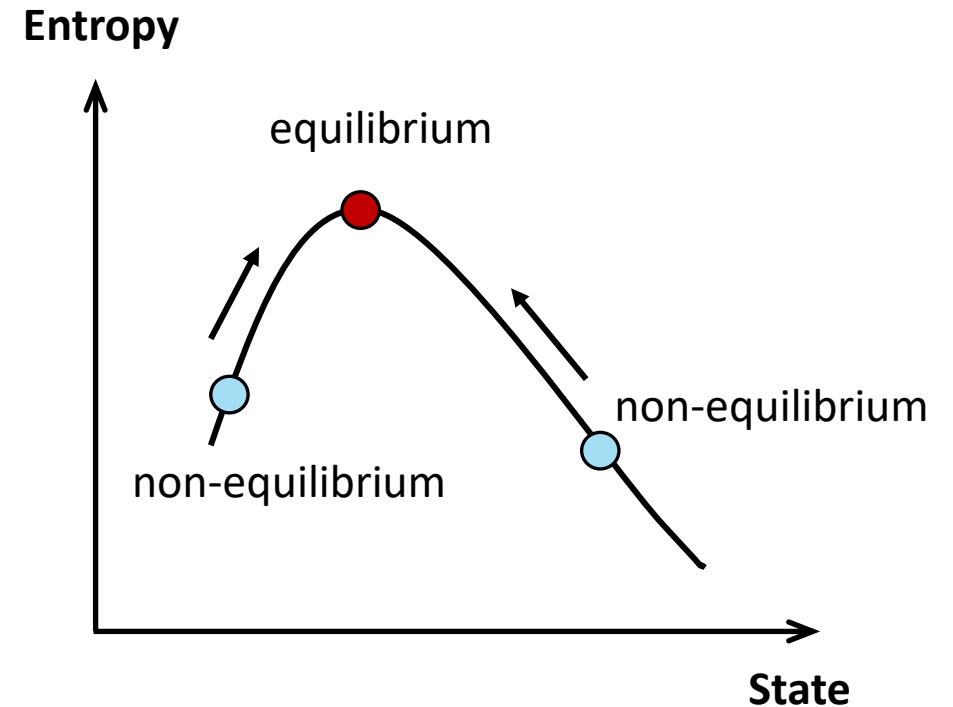
Method of Thermodynamic Stability

- **Lyapunov functional for hydro-eqs.**
  (positive and non-increasing)

$$E = \Psi_{\text{eq}} - \Psi_{\text{non-eq}} = \int d\Sigma E^\mu n_\mu$$

$$E^\mu = -\delta s^\mu_{\text{fluid}} - \alpha_I \delta J^{I,\mu}_{\text{fluid}}$$

- **Criteria for thermodynamic stability**

(i) $E^\mu n_\mu \geq 0$ for any n^μ with $n_0 > 0, n^\mu n_\mu = 1$,

(ii) $E^\mu n_\mu = 0$ if and only if all perturbations are zero,
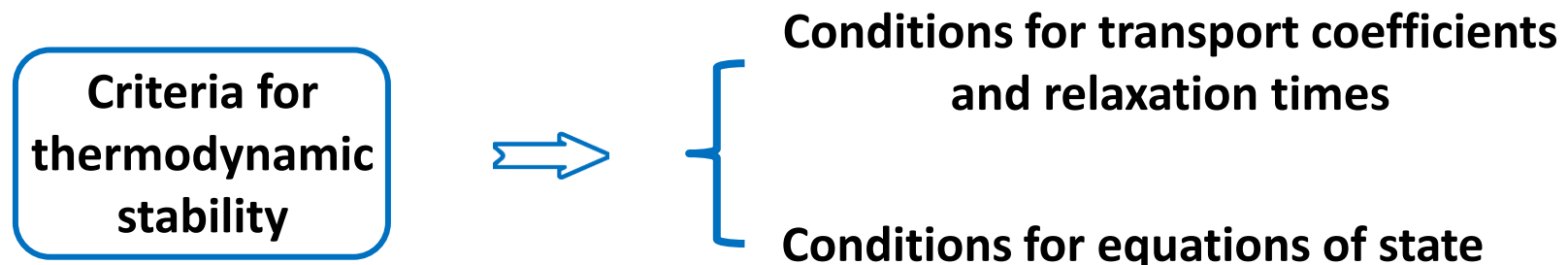
(iii) $\partial_\mu E^\mu \leq 0$.

Gavassino, Antonelli, Haskell, PRL 128, 010606 (2022)

Thermodynamic Stability of Spin hydrodynamics

➤ **In spin hydrodynamics we have**

$$E^\mu = -\delta s^\mu + \frac{u_\nu}{T}\delta\Theta^{\mu\nu} - \frac{1}{T}\omega_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma}$$

➤ **Conditions for thermodynamic stability of spin-hydro.**

Criteria for thermodynamic stability ⟹

Conditions for transport coefficients and relaxation times

Conditions for equations of state

Ren, Yang, DLW, Pu, PRD 110, 034010 (2024)
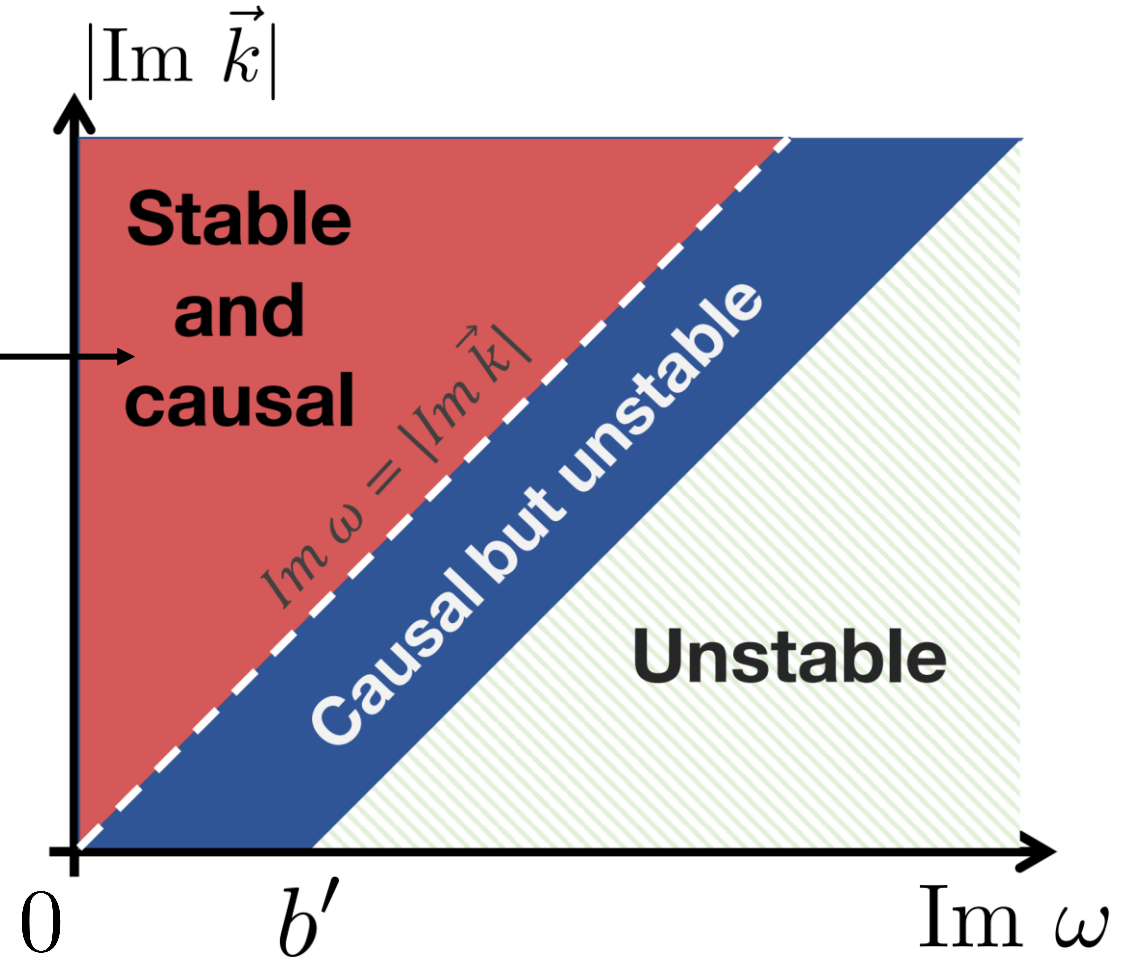
Causal, Stable, and Well-posed

> **Linearized spin hydrodynamic equations can be written in a "standard form"**

$$E^{\mu}_{AB}\partial_{\mu}\varphi^{B} = -\sigma_{AB}\varphi^{B} - \Xi_{[AB]}\varphi^{B}$$

spin contribution

Equations are symmetric-hyperbolic

Equations are **well-posed**

DLW, Pu, PRD 109, L031504 (2024)

Summary

Summary

- ➢ **There exists a regime where the spin density decays slowly, allowing it to be treated as a hydrodynamic variable.**

- ➢ **In Bjorken flow, we can find a late-time attractor solution for the decay rate of spin density.**

- ➢ **The spin hydrodynamic equations with nonzero relaxation terms is causal, stable, and well-posed near equilibrium.**

Thank you!

Backup

Constitutive Relations from Entropy Principle

First order constitutive relations:

$$h^\mu - \frac{e+p}{n}\nu^\mu = \kappa[\Delta^{\mu\nu}\partial_\nu T - T(u\cdot\partial)u^\mu],$$
$$\pi^{\mu\nu} = \eta_s\partial^{<\mu}u^{\nu>} + \zeta(\partial\cdot u)\Delta^{\mu\nu},$$
$$q^\mu = \lambda\left[\frac{1}{T}\Delta^{\mu\nu}\partial_\nu T + (u\cdot\partial)u^\mu - 4\omega^{\mu\nu}u_\nu\right],$$
$$\phi^{\mu\nu} = -\gamma\left(\Omega^{\mu\nu} - \frac{2}{T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta}\right),$$

entropy principle

$$\partial_\mu \mathcal{S}^\mu \geq 0$$

Entropy current:

$$\mathcal{S}^\mu = su^\mu + \frac{1}{T}h^\mu + \frac{1}{T}q^\mu - \frac{\mu}{T}\nu^\mu$$

$$\Omega^{\mu\nu} \equiv -\Delta^{\mu\rho}\Delta^{\nu\sigma}\partial_{[\rho}(u_{\sigma]}/T) \qquad \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$$

$$\Theta^{\mu\nu} = (e+p)u^\mu u^\nu - pg^{\mu\nu} + 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} + \Theta^{[\mu\nu]},$$
$$J^{\lambda\mu\nu} = x^\mu\Theta^{\lambda\nu} - x^\nu\Theta^{\lambda\mu} + \Sigma^{\lambda\mu\nu},$$
$$j^\mu = nu^\mu + \nu^\mu,$$

$$\Theta^{[\mu\nu]} = 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}, \quad \Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma_\perp^{\lambda\mu\nu}.$$
$$\sim \mathcal{O}(\partial)$$

Florkowski, Ryblewsk, Kumar, PPNP (2019)
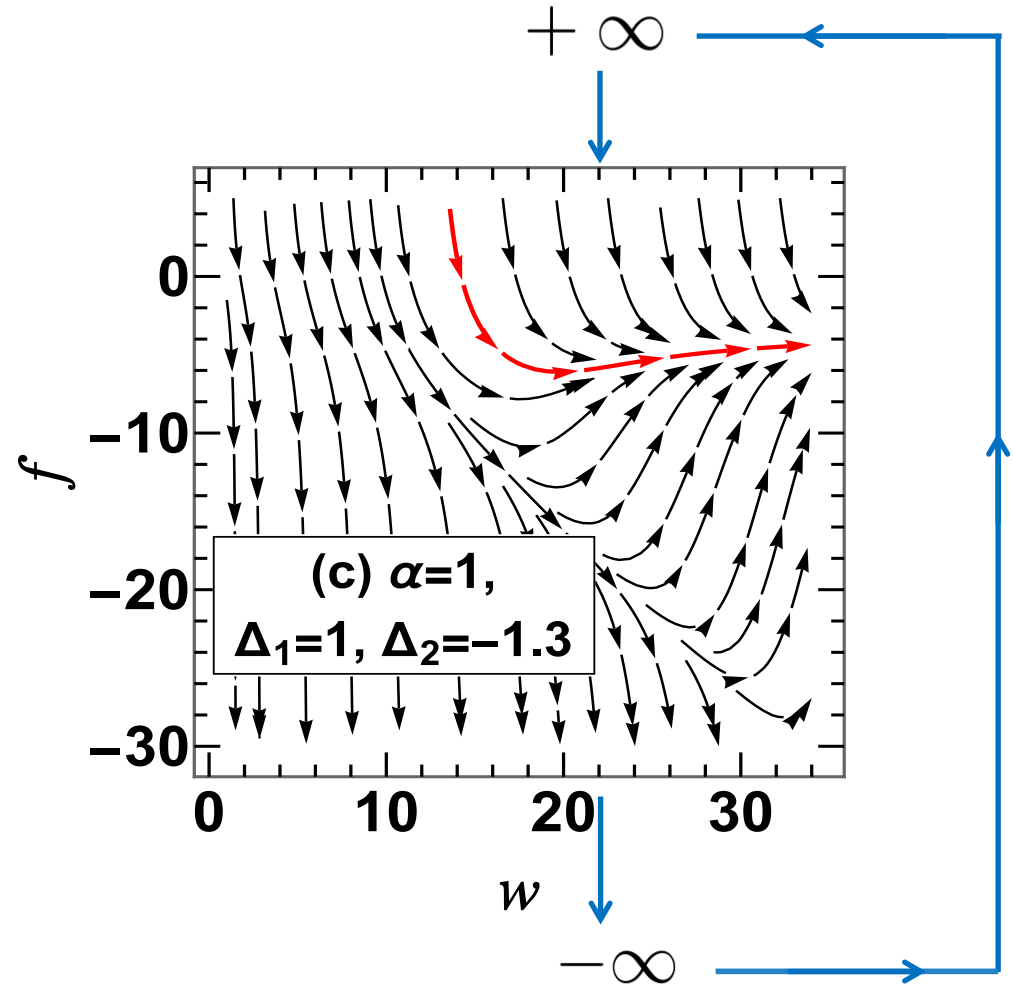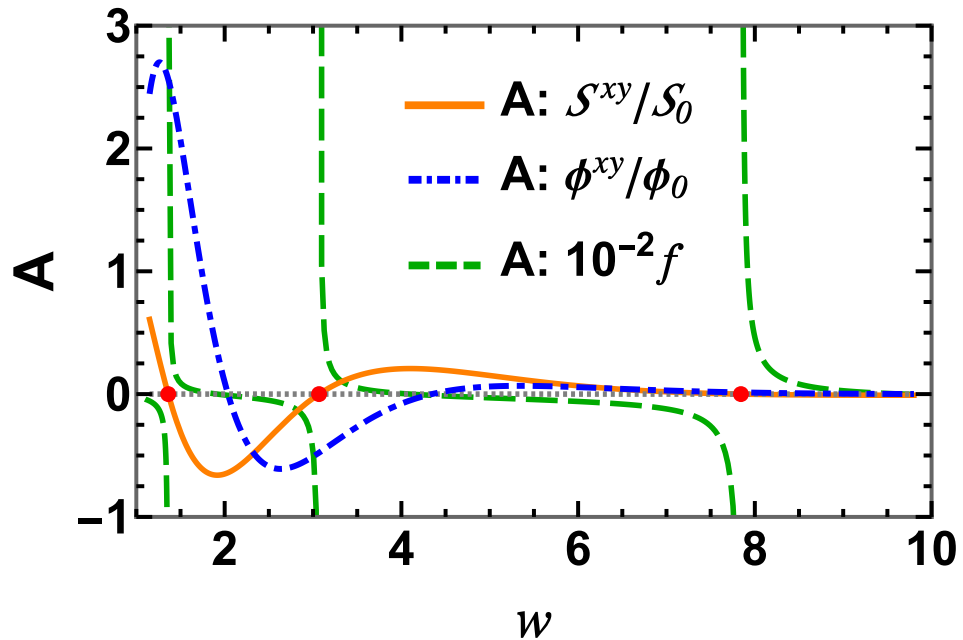Hattori, Hongo, Huang, Matsuo, Taya, PLB (2019)
Hongo, Huang, Kaminski, Stephanov, Yee, JHEP (2021)
Fukushima, Pu, PLB (2021)

No Singularity for Spin Density

$$f(w) \equiv \Delta_1 \frac{w}{S^{xy}} \frac{dS^{xy}}{dw} = \frac{\tau}{S^{xy}} \frac{dS^{xy}}{d\tau}$$

$$S^{xy} \to 0, \quad f \to \pm\infty$$

Legend:
- A: S^{xy}/S_0
- A: ϕ^{xy}/ϕ_0
- A: $10^{-2}f$

(c) $\alpha=1$, $\Delta_1=1$, $\Delta_2=-1.3$

Why Late-Time Attractors Exist

➤ **Assume that the magnitude of γ small**

$$\tau_\phi \Delta^{\mu\alpha} \Delta^{\nu\beta} u^\rho \nabla_\rho \phi_{\alpha\beta} + \phi^{\mu\nu} = 2\gamma \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_{[\alpha} u_{\beta]} + 2\omega_{\alpha\beta})$$

➤ **The spin source term decays exponentially**

$$\phi^{xy} \approx \phi_0 \exp\left(-\frac{w}{\Delta_1}\right) \to 0$$

➤ **Total spin is conserved approximately**

power-law decay

$$\partial_\lambda \Sigma^{\lambda xy} \approx 0 \implies \frac{dS^{xy}}{d\tau} + \frac{1}{\tau} S^{xy} \approx 0 \implies \boxed{S^{xy} \approx S_0 \frac{\tau_1}{\tau}}$$

Necessary Criterion for Causality

➢ **Suppose that the perturbations have compact support at t = 0.**

➢ **If propagation speed is subluminal, the necessary criterion is**

$$\lim_{|\vec{k}| \to +\infty} \left\{ \frac{|\mathrm{Re}\ \omega|}{|\vec{k}|} \leq 1, \ |\omega/\vec{k}| \text{ is bounded} \right\}, \ \vec{k} \in \mathbb{R}^3$$

➢ **Dispersion relation of non-relativistic diffusion:** $\omega \propto k^2$ **acausal!**

E. Krotscheck, W. Kundt, Communications in Mathematical Physics 60, 171 (1978)
P. D. Lax, Hyperbolic Partial Differential Equations (2006)

Covariant Stability Implies Causality

➤ **Criterion for covariant stability**

Heller, Serantes, Spalinski, Withers, PRL (2023)
Gavassino, PLB (2023)

$$\text{Im } \omega \leq |\text{Im } \vec{k}|, \quad \text{for } \vec{k} \in \mathbb{C}^3$$

➤ **Improved criterion for causality**

DLW, Pu, PRD 109, L031504 (2024)

$$\text{Im } \omega \leq |\text{Im } \vec{k}| + b', \text{ for } \vec{k} \in \mathbb{C}^3$$