Attractor Behavior and Thermodynamic Stability in Spin Hydrodynamics

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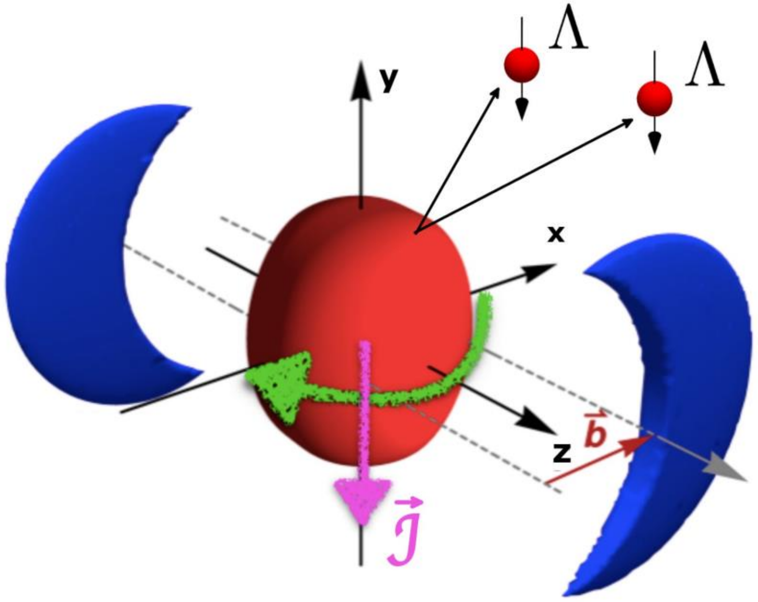
Outline

- Introduction
- Evolution of spin density in spin hydrodynamics
- Causality and stability analysis for spin Hydrodynamics
- Summary

Introduction

Rotation and Polarization



picture from Florkowski, Kumar, Ryblewski, PPNP (2019)

- **Large angular momentum in noncentral heavy ion collisions**

$$J \sim \frac{A\sqrt{s}}{2} b \sim 10^5 \hbar$$

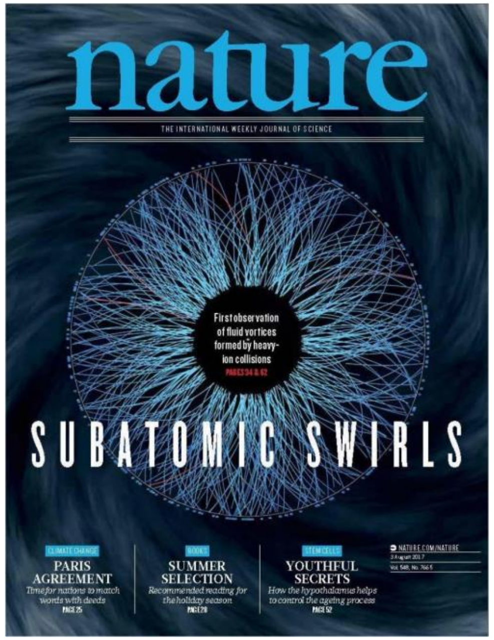
Vorticity of quark gluon plasma

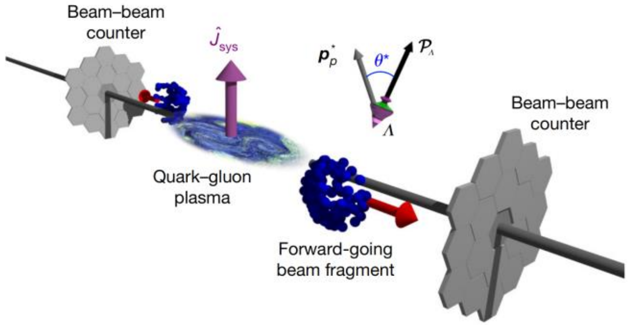
⇓ **(Total angular momentum conservation)**

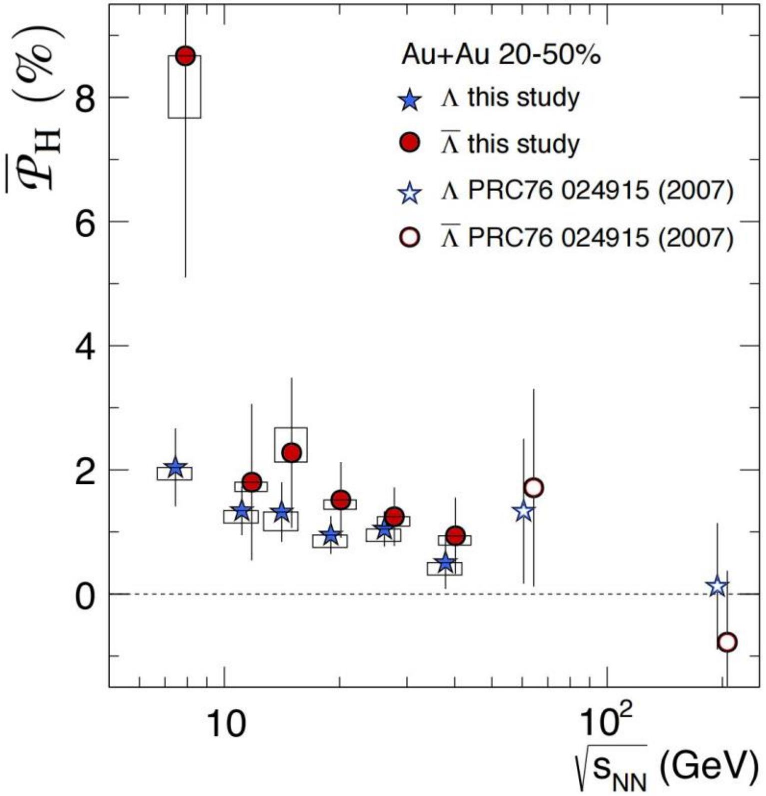
Polarization of particles along vorticity

Liang, Wang, PRL (2005); PLB (2005)
Gao, Chen, Deng, Liang, Wang, Wang, PRC (2008)

Most Vortical Fluid and Spin Polarization







Adamczyk et al. (STAR), Nature 548, 62 (2017)

- **Weak decay:** $\Lambda \to p + \pi^-$
- **Proton distribution:**

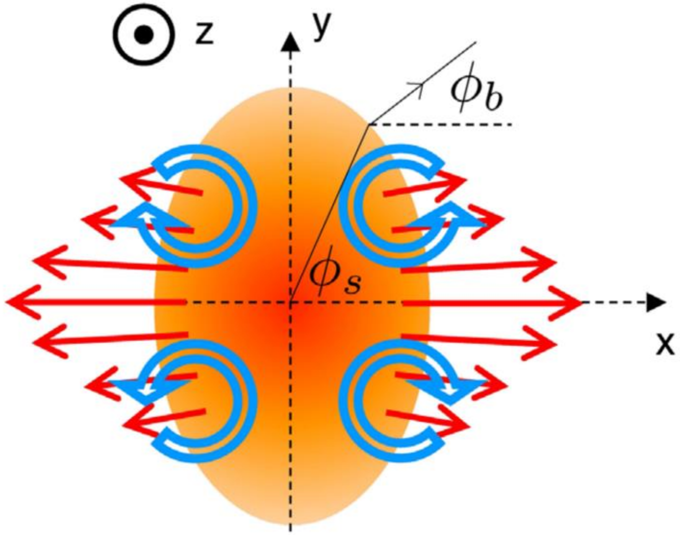
$$\frac{dN}{d\cos\theta^*} = \frac{1}{2}(1 + \alpha_{\rm H}|\vec{\mathcal{P}}_{\rm H}|\cos\theta^*)$$

- **Large vorticity:**

$$\omega \approx (9 \pm 1) \times 10^{21}\,{\rm s}^{-1}$$

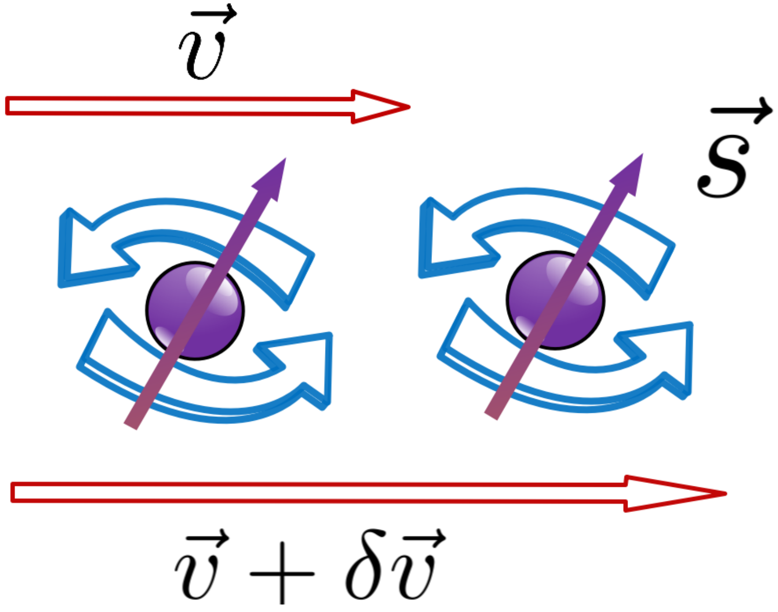
Most vortical fluid!

Spin-Orbit Coupling in Hydrodynamics



J. Adam et al. (STAR), PRL. 123, 132301 (2019)

$\Longleftrightarrow$



How to study the transform between spin and orbital angular momentum at macroscopic level?

Relativistic Spin Hydrodynamics

Modified thermodynamic relations:

$$e + p = Ts + \mu n + \underline{\omega_{\mu\nu} S^{\mu\nu}}$$
$$de = Tds + \mu dn + \underline{\omega_{\mu\nu} dS^{\mu\nu}}$$

energy density: e pressure: p

temperature: T entropy density: s

spin density: $S^{\mu\nu}$ spin chemical potential: $\omega_{\mu\nu}$

Florkowski, Kumar, Ryblewsk, PPNP (2019)
Hattori, Hongo, Huang, Matsuo, Taya, PLB (2019)
Hongo, Huang, Kaminski, Stephanov, Yee, JHEP (2021)
Fukushima, Pu, PLB (2021)

Conservation equations:

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0$$

↓ energy-momentum ↓ particle number ↓ total angular momentum

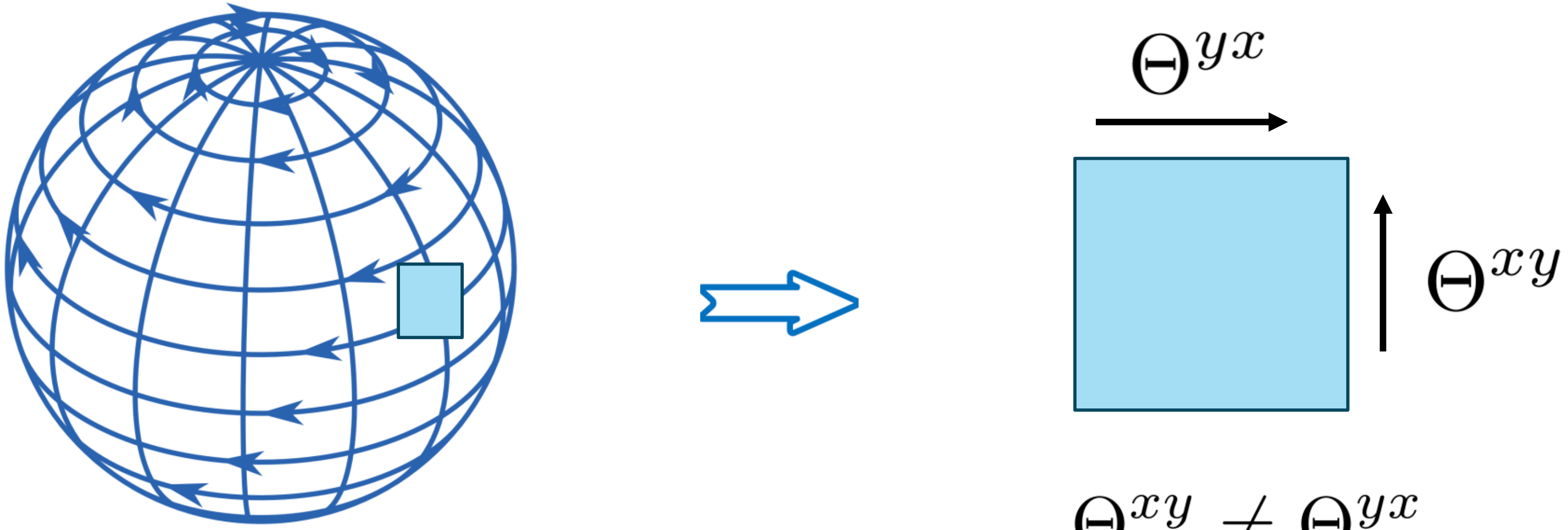
$$J^{\lambda\mu\nu} = x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu} + \Sigma^{\lambda\mu\nu}$$
$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma_\perp^{\lambda\mu\nu}$$

↓

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$$

Convertibility between spin and orbital angular momentum

Antisymmetric Part of EMT



$$\Theta^{xy} \neq \Theta^{yx}$$

There exists nonzero net torque acting on the fluid cell

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$$

Problems in Spin Hydrodynamics

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0$$

$$h^\mu - \frac{e+p}{n}\nu^\mu = \kappa[\Delta^{\mu\nu}\partial_\nu T - T(u\cdot\partial)u^\mu],$$
$$\pi^{\mu\nu} = \eta_s \partial^{<\mu}u^{\nu>} + \zeta(\partial\cdot u)\Delta^{\mu\nu},$$
$$q^\mu = \lambda\left[\frac{1}{T}\Delta^{\mu\nu}\partial_\nu T + (u\cdot\partial)u^\mu - 4\omega^{\mu\nu}u_\nu\right],$$
$$\phi^{\mu\nu} = -\gamma\left(\Omega^{\mu\nu} - \frac{2}{T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta}\right),$$

Florkowski, Ryblewsk, Kumar, PPNP (2019)
Hattori, Hongo, Huang, Matsuo, Taya, PLB (2019)
Hongo, Huang, Kaminski, Stephanov, Yee, JHEP (2021)
Fukushima, Pu, PLB (2021)

1. **Can spin density be considered a hydrodynamic variable?**
2. **Initial conditions for spin hydrodynamics are still unknown.**
3. **How to ensure causality and stability in spin hydrodynamics?**
4. **….**

Systematic studies for spin hydrodynamics are required!

Evolution of Spin Density in Spin Hydrodynamics

- Regime for spin hydrodynamics
- Late-time attractor

Spin Hydrodynamics in Bjorken Flow

$$\partial_\mu \Theta^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0$$

$$e = 3p$$

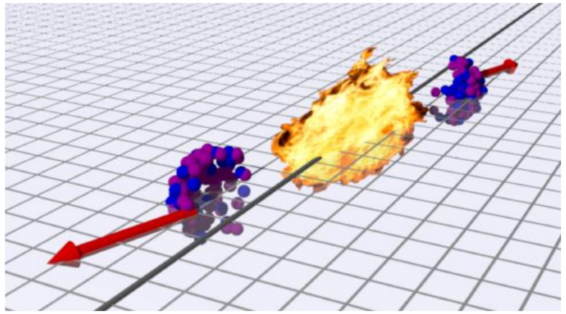
$$S^{\mu\nu} = aT^2\omega^{\mu\nu}$$

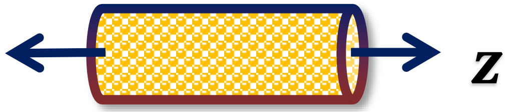
$$n \sim 0$$

$$u^\mu = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

J. D. Bjorken, Phys. Rev. D 27, 140(1983)

picture from Becattini, Lisa, ARNPS (2020)





- **Symmetric constraints expressed in Lie derivative**

$$\mathcal{L}_\kappa S^{\mu\nu} = 0 \qquad \kappa \in \{\partial_x,\ \partial_y,\ x\partial_y - y\partial_x,\ z\partial_t + t\partial_z\}$$

- **The hydrodynamic equations in a Bjorken flow are self-consistent iff**

$$S^{\mu\nu} = 0 \text{ for } (\mu\nu) \neq (xy)$$

DLW, Fang, Pu, PRD 104, 11404 (2021)
DLW, Yan, Pu, arXiv: 2408.03781

Asymptotic Analysis for Spin Hydrodynamics

DLW, Yan, Pu, arXiv: 2408.03781

- **Parameterize time-dependent coefficients:** $\mathrm{Kn}^{-1} \approx w \equiv \frac{\tau}{\tau_\phi} = \left(\frac{\tau}{\tau_1}\right)^{\Delta_1}, \quad \frac{\tau_\phi \gamma}{\chi} = \alpha w^{\Delta_2}$

- **Late-time asymptotic solutions for $\Delta_1 > 0$:**

$\Delta_2 = -1$	$S_{(1)} \propto e^{-w/\Delta_1}, S_{(2)} \sim w^{-(1+8\alpha)/\Delta_1}$
$\Delta_2 < -1$	$S_{(1)} \propto e^{-w/\Delta_1}, S_{(2)} \sim w^{-1/\Delta_1}$

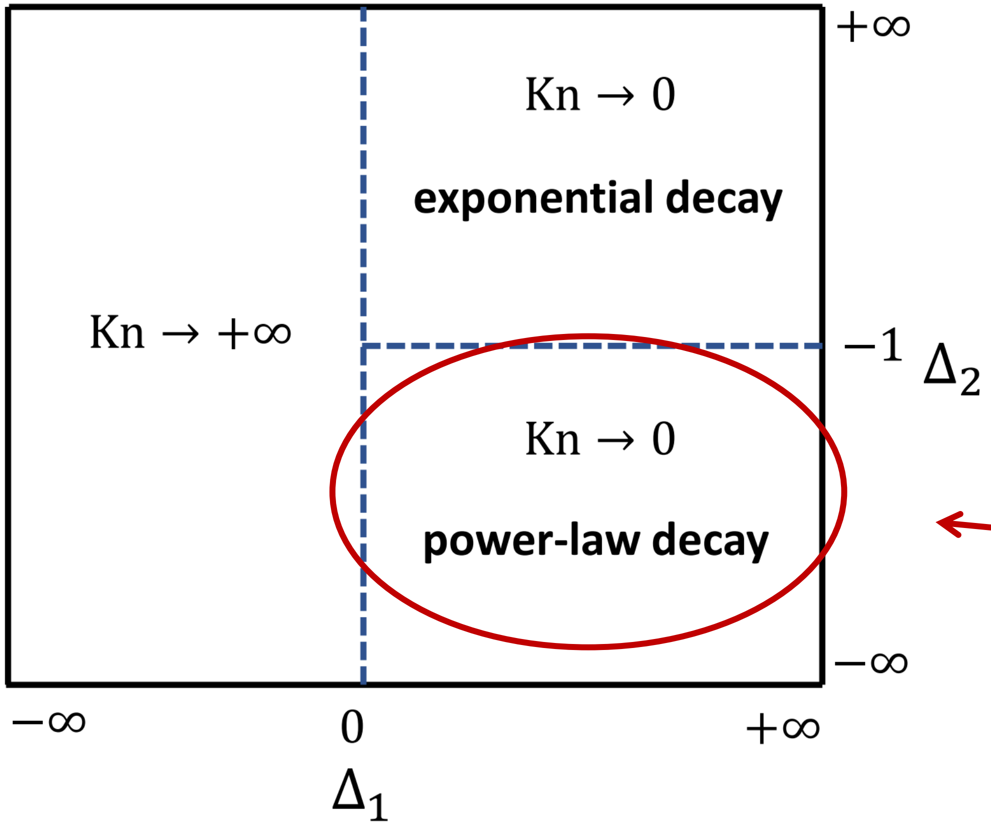
power-law decay (hydro-mode)

$\Delta_2 = 0$	$S_{(1),(2)} \propto e^{-w(1\pm\sqrt{1-32\alpha})/(2\Delta_1)}$
$-1 < \Delta_2 < 0$	$S_{(1)} \propto e^{-w/\Delta_1}, S_{(2)} \propto \exp\left[-\frac{8\alpha w^{1+\Delta_2}}{\Delta_1(1+\Delta_2)}\right]$
$\Delta_2 = -1$	$S_{(1)} \propto e^{-w/\Delta_1}, S_{(2)} \sim w^{-(1+8\alpha)/\Delta_1}$

exponential decay (non-hydro-mode)

Regime for Spin Hydrodynamics

DLW, Yan, Pu, arXiv: 2408.03781



$$\mathrm{Kn}^{-1} \approx w \equiv \frac{\tau}{\tau_\phi} = \left(\frac{\tau}{\tau_1}\right)^{\Delta_1}, \quad \frac{\tau_\phi \gamma}{\chi} = \alpha w^{\Delta_2}$$

Spin hydrodynamics is well-defined in this regime!

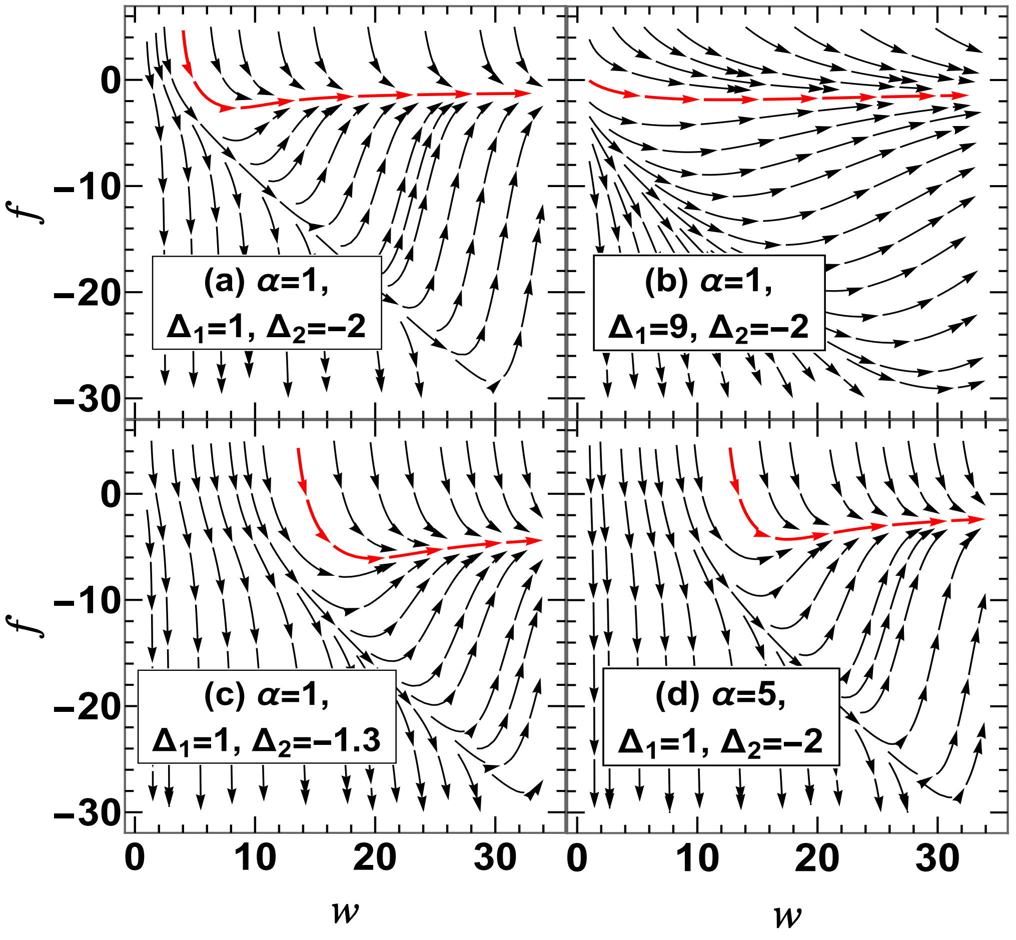
Late-Time Attractor Behavior

- **Decay rate of spin density:**

$$f(w) \equiv \Delta_1 \frac{w}{S^{xy}} \frac{dS^{xy}}{dw} = \frac{\tau}{S^{xy}} \frac{dS^{xy}}{d\tau}$$

- **Attractor solution:**

$$f \to \begin{cases} -1 - 8\alpha, & \Delta_2 = -1, \\ -1, & \Delta_2 < -1. \end{cases}$$



Late-time decay rate is
insensitive to initial conditions!

DLW, Yan, Pu, arXiv: 2408.03781

Why Late-Time Attractors Exist

➢ **Assume that the magnitude of γ small**

$$\tau_\phi \Delta^{\mu\alpha}\Delta^{\nu\beta}u^\rho\nabla_\rho\phi_{\alpha\beta} + \phi^{\mu\nu} = 2\gamma\Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_{[\alpha}u_{\beta]} + 2\omega_{\alpha\beta})$$

➢ **The spin source term decays exponentially**

$$\phi^{xy} \approx \phi_0 \exp\left(-\frac{w}{\Delta_1}\right) \to 0$$

➢ **Total spin is conserved approximately**

power-law decay

$$\partial_\lambda \Sigma^{\lambda xy} \approx 0 \quad \Rightarrow \quad \frac{dS^{xy}}{d\tau} + \frac{1}{\tau}S^{xy} \approx 0 \quad \Rightarrow \quad S^{xy} \approx S_0\frac{\tau_1}{\tau}$$

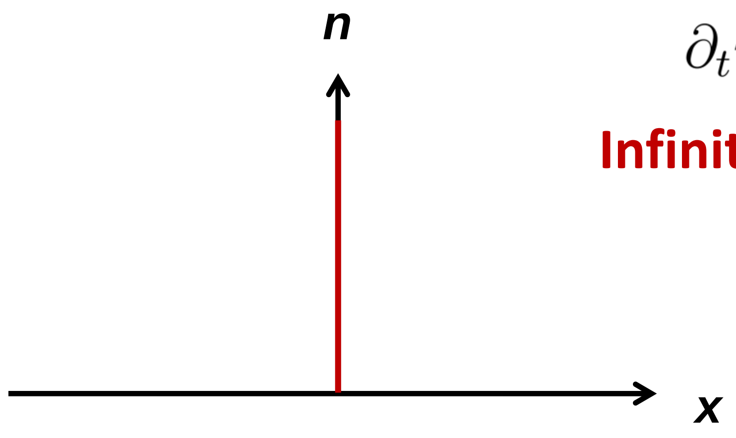
Causality and Stability Analysis for Spin Hydrodynamics

- **Acausal modes in first order spin hydrodynamics**
- **Remove acausal modes**
- **Thermodynamic stability analysis**

Example: Non-relativistic Diffusion

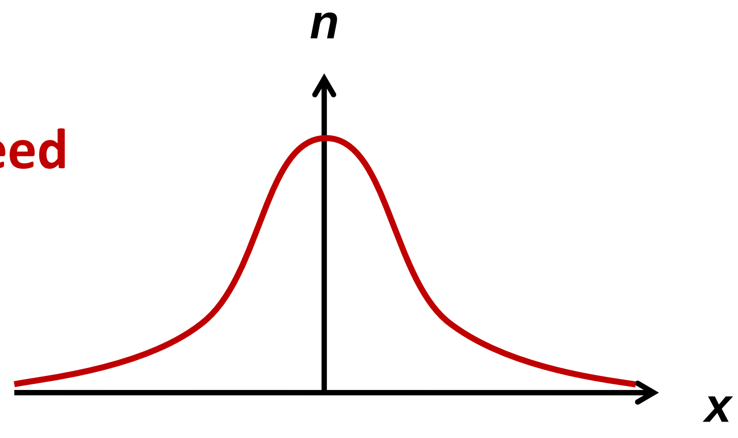
$$n(t=0,x)=\delta(x)$$

$$n(t>0,x)=\frac{1}{2\sqrt{\pi D_n t}}\exp\left(-\frac{x^2}{4D_n t}\right)$$



$$\partial_t n - D_n \partial_x^2 n = 0$$

Infinite propagation speed



- The initial data $n(0,x)$ is zero for $x \neq 0$
- For any small time $t>0$, $n(t,x)$ is nonzero everywhere

Acausal Modes in First Order Spin Hydrodynamics

- **Independent small perturbations on top of equilibrium satisfy**

$$\partial_\mu \delta\Theta^{\mu\nu} = 0, \quad \partial_\lambda \delta J^{\lambda\mu\nu} = 0, \quad \partial_\mu \delta j^\mu = 0.$$

- **One mode behaves like non-relativistic diffusion**

$$\omega \propto k^2$$

$$\partial_t n - D_n \partial_x^2 n = 0$$

$$\Downarrow$$

$$\omega = -i D_n k^2$$

Infinite propagation speed!

Krotscheck, Kundt, Communications in Mathematical Physics 60, 171 (1978)
Xie, DLW, Yang, Pu, PRD 108, 094031 (2023); DLW, Pu, PRD 109, L031504 (2024)

Remove Modes With Infinite Propagation Speed

- **Introduce nonzero relaxation times:**

$$\begin{aligned}
\tau_q \Delta^{\mu\nu} \frac{d}{d\tau} q_\nu + q^\mu &= \lambda(T^{-1}\Delta^{\mu\alpha}\partial_\alpha T + Du^\mu - 4\omega^{\mu\nu}u_\nu), \\
\tau_\phi \Delta^{\mu\alpha}\Delta^{\nu\beta} \frac{d}{d\tau}\phi_{\alpha\beta} + \phi^{\mu\nu} &= 2\gamma_s \Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_{[\alpha}u_{\beta]} + 2\omega_{\alpha\beta}), \\
\tau_\pi \Delta^{\alpha<\mu}\Delta^{\nu>\beta} \frac{d}{d\tau}\pi_{\alpha\beta} + \pi^{\mu\nu} &= 2\eta\partial^{<\mu}u^{\nu>}, \\
\tau_\Pi \frac{d}{d\tau}\Pi + \Pi &= -\zeta\partial_\mu u^\mu,
\end{aligned}$$

Liu, Huang, Nucl. Sci. Tech. (2020)
Xie, DLW, Yang, Pu, PRD 108, 094031 (2023)

- **The thermodynamic flux cannot instantaneously vanish/appear when thermodynamic force is suddenly switched off/on.**

Israel, Stewart, Annals Phys. (1979); Muronga, PRC (2004); Koide, Denicol, Mota, Kodama, PRC (2007)

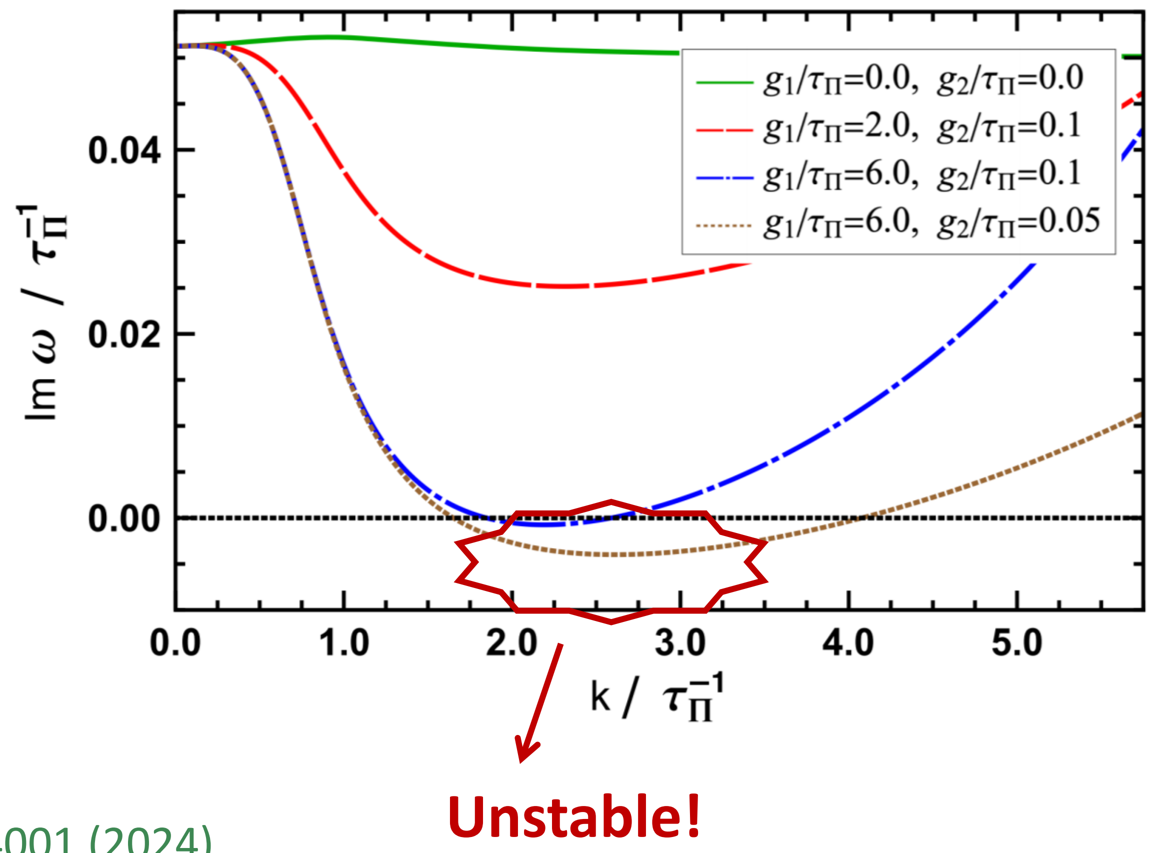
Is Equilibrium State Unstable?

- **Check stability by linear mode analysis**

$$\delta\varphi \sim e^{i\omega t - ikx}, \quad \mathrm{Im}\ \omega(k) > 0$$

- **Complicated dispersion relations.**

- **Necessary conditions are obtainable, but they are insufficient for stability.**



Xie, DLW, Yang, Pu, PRD 108, 094031 (2023)
Daher, Florkowski, Ryblewski, Taghinavaz, PRD 109, 114001 (2024)

Method of Thermodynamic Stability

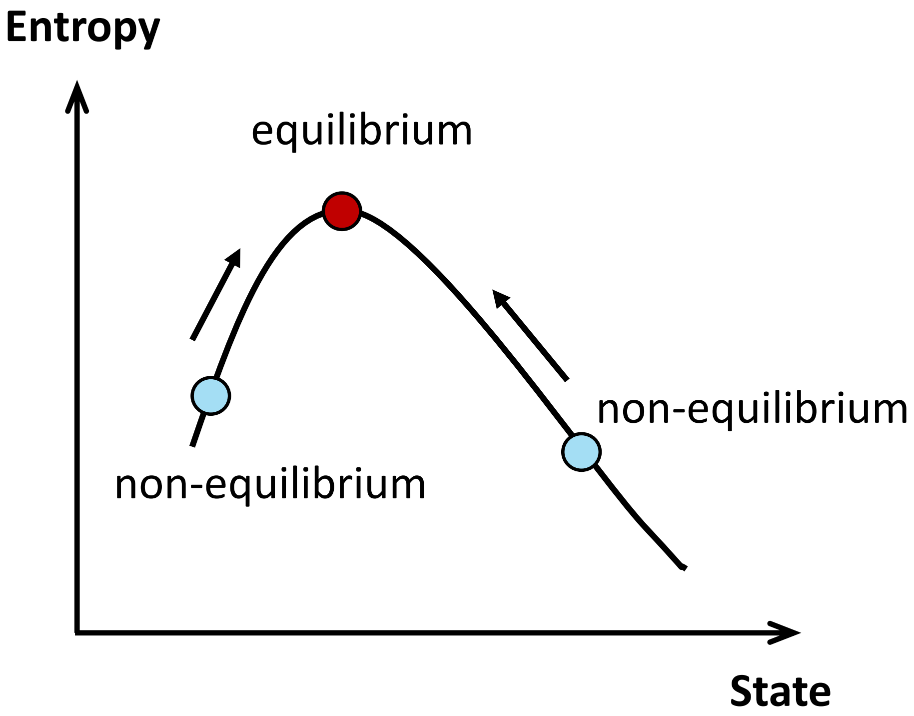
➢ **Entropy principle:**

$$\Delta S_{\text{bath}} = \alpha_I \Delta Q^I_{\text{fluid}}$$

$$\Delta S_{\text{fluid}} + \Delta S_{\text{bath}} \geq 0$$

$$\Psi = S_{\text{fluid}} + \alpha_I Q^I_{\text{fluid}}$$

Ψ { **maximized in equilibrium state**
non-decreasing functional }



Gavassino, Antonelli, Haskell, PRL 128, 010606 (2022)

Method of Thermodynamic Stability

- **Lyapunov functional for hydro-eqs.**
  (positive and non-increasing)

$$E = \Psi_{\text{eq}} - \Psi_{\text{non-eq}} = \int d\Sigma E^{\mu} n_{\mu}$$

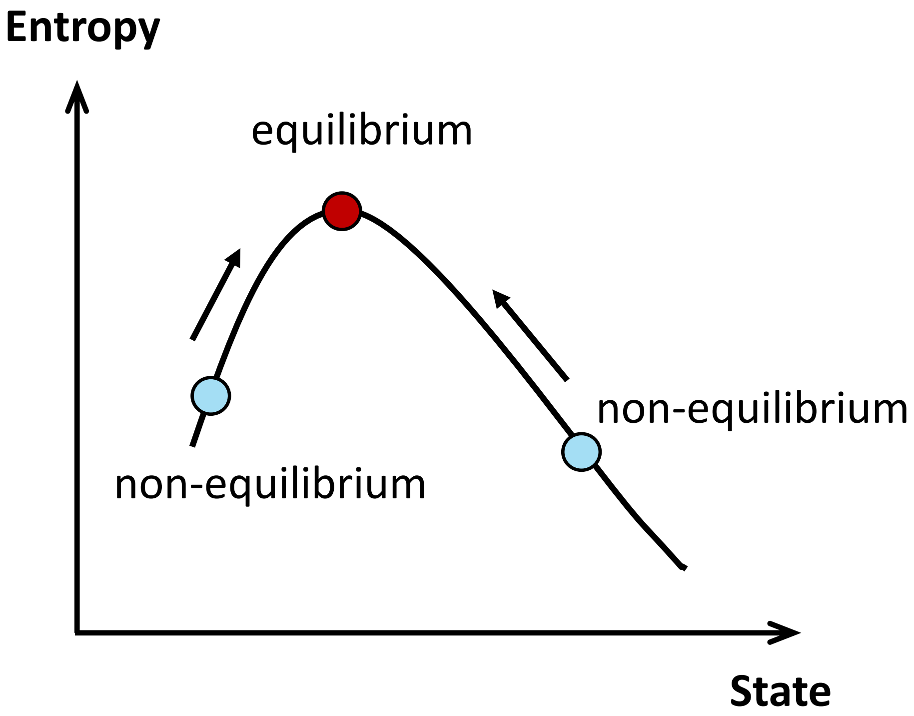
$$E^{\mu} = -\delta s^{\mu}_{\text{fluid}} - \alpha_I \delta J^{I,\mu}_{\text{fluid}}$$

- **Criteria for thermodynamic stability**

(i) $\quad E^{\mu} n_{\mu} \geq 0$ for any n^{μ} with $n_0 > 0, n^{\mu} n_{\mu} = 1,$

(ii) $\quad E^{\mu} n_{\mu} = 0$ if and only if all perturbations are zero,

(iii) $\quad \partial_{\mu} E^{\mu} \leq 0.$



Gavassino, Antonelli, Haskell, PRL 128, 010606 (2022)

Thermodynamic Stability of Spin hydrodynamics

- **In spin hydrodynamics we have**

$$E^{\mu} = -\delta s^{\mu} + \frac{u_{\nu}}{T}\delta\Theta^{\mu\nu} - \frac{1}{T}\omega_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma}$$

- **Conditions for thermodynamic stability of spin-hydro.**

Criteria for thermodynamic stability ⟹

- **Conditions for transport coefficients and relaxation times**
- **Conditions for equations of state**

Ren, Yang, DLW, Pu, PRD 110, 034010 (2024)

Causal, Stable, and Well-posed

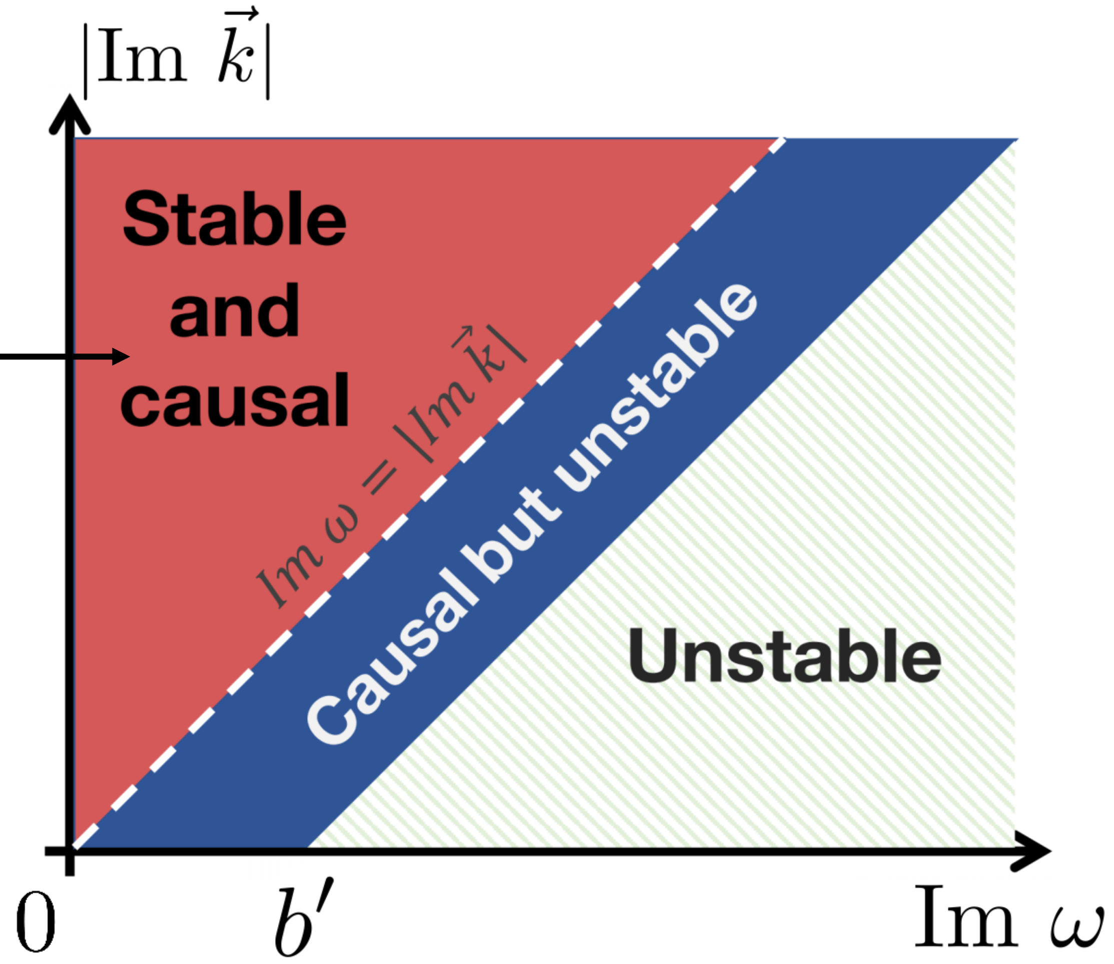
➢ **Linearized spin hydrodynamic equations can be written in a "standard form"**

$$E^{\mu}_{AB}\partial_\mu \varphi^B = -\sigma_{AB}\varphi^B - \Xi_{[AB]}\varphi^B$$

spin contribution

Equations are symmetric-hyperbolic

Equations are **well-posed**



DLW, Pu, PRD 109, L031504 (2024)

Summary

Summary

- **There exists a regime where the spin density decays slowly, allowing it to be treated as a hydrodynamic variable.**
- **In Bjorken flow, we can find a late-time attractor solution for the decay rate of spin density.**
- **The spin hydrodynamic equations with nonzero relaxation terms is causal, stable, and well-posed near equilibrium.**

Thank you!

Backup

Constitutive Relations from Entropy Principle

First order constitutive relations:

$$h^\mu - \frac{e+p}{n}\nu^\mu = \kappa[\Delta^{\mu\nu}\partial_\nu T - T(u\cdot\partial)u^\mu],$$
$$\pi^{\mu\nu} = \eta_s \partial^{<\mu}u^{\nu>} + \zeta(\partial\cdot u)\Delta^{\mu\nu},$$
$$q^\mu = \lambda\left[\frac{1}{T}\Delta^{\mu\nu}\partial_\nu T + (u\cdot\partial)u^\mu - 4\omega^{\mu\nu}u_\nu\right],$$
$$\phi^{\mu\nu} = -\gamma\left(\Omega^{\mu\nu} - \frac{2}{T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta}\right),$$

entropy principle ⟸

$$\partial_\mu \mathcal{S}^\mu \geq 0$$

Entropy current:

$$\mathcal{S}^\mu = su^\mu + \frac{1}{T}h^\mu + \frac{1}{T}q^\mu - \frac{\mu}{T}\nu^\mu$$

$$\Omega^{\mu\nu} \equiv -\Delta^{\mu\rho}\Delta^{\nu\sigma}\partial_{[\rho}(u_{\sigma]}/T) \qquad \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$$

$$\Theta^{\mu\nu} = (e+p)u^\mu u^\nu - pg^{\mu\nu} + 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} + \Theta^{[\mu\nu]},$$
$$J^{\lambda\mu\nu} = x^\mu\Theta^{\lambda\nu} - x^\nu\Theta^{\lambda\mu} + \Sigma^{\lambda\mu\nu},$$
$$j^\mu = nu^\mu + \nu^\mu,$$

$$\Theta^{[\mu\nu]} = 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}, \quad \Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \Sigma_\perp^{\lambda\mu\nu}.$$

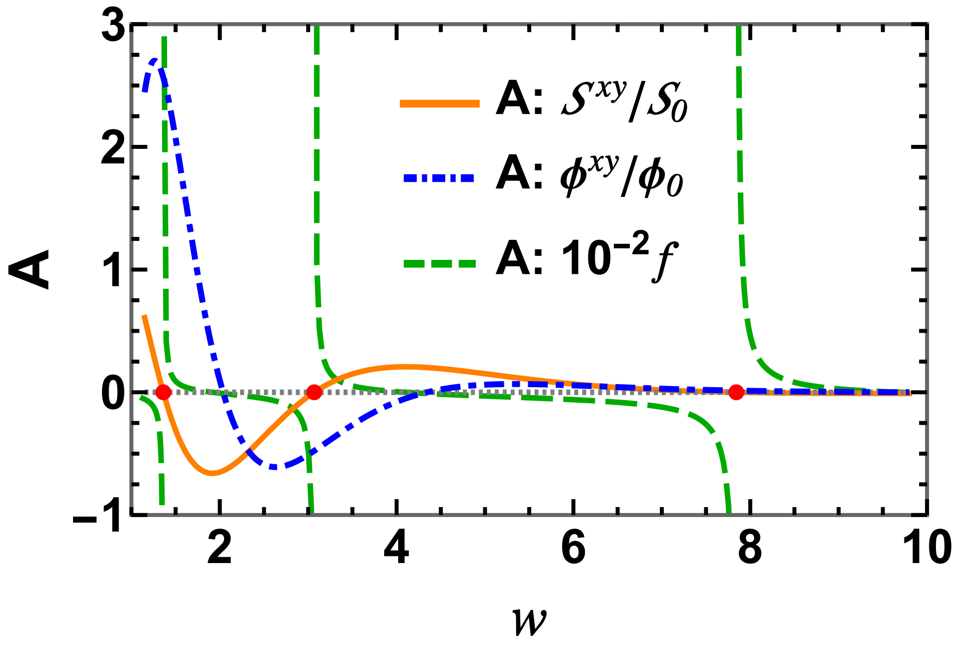
$$\sim \mathcal{O}(\partial)$$

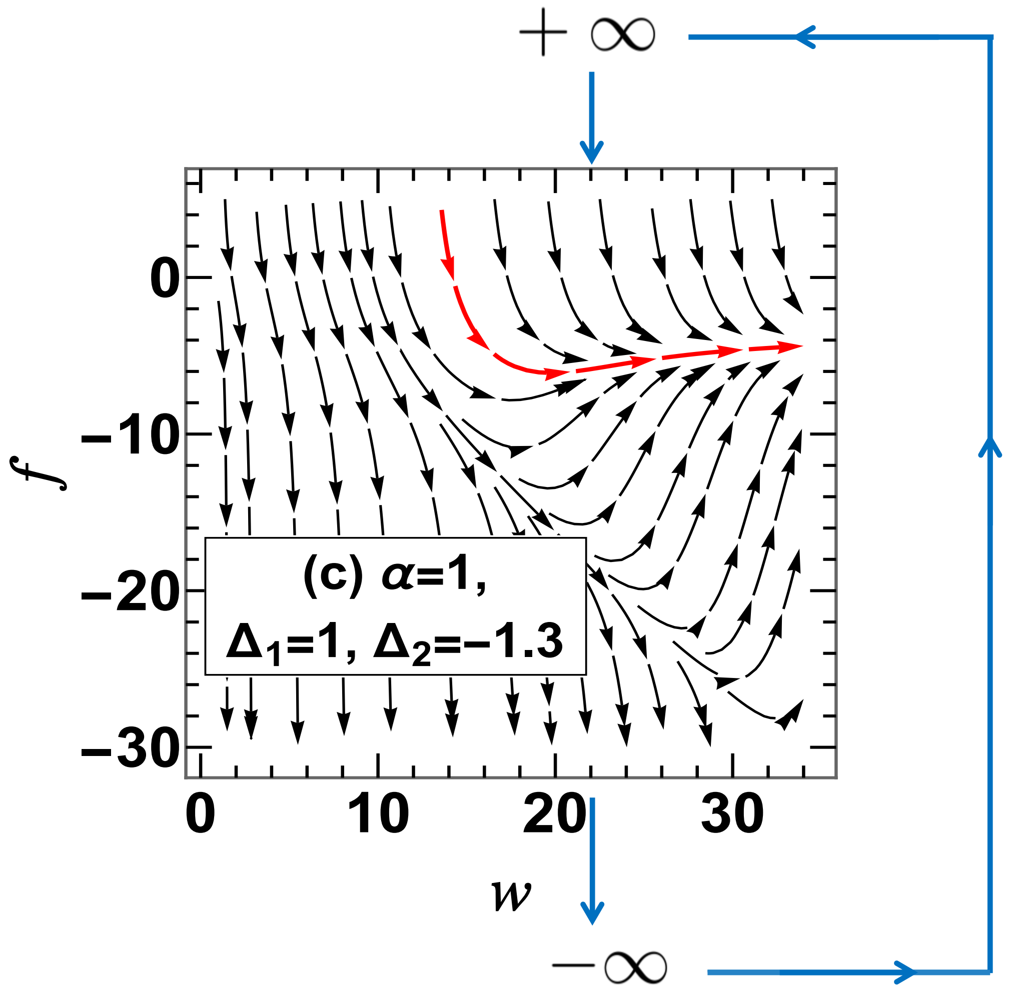
Florkowski, Ryblewsk, Kumar, PPNP (2019)
Hattori, Hongo, Huang, Matsuo, Taya, PLB (2019)
Hongo, Huang, Kaminski, Stephanov, Yee, JHEP (2021)
Fukushima, Pu, PLB (2021)

No Singularity for Spin Density

$$f(w) \equiv \Delta_1 \frac{w}{S^{xy}} \frac{dS^{xy}}{dw} = \frac{\tau}{S^{xy}} \frac{dS^{xy}}{d\tau}$$

$$S^{xy} \to 0, \quad f \to \pm\infty$$





Necessary Criterion for Causality

- **Suppose that the perturbations have compact support at t = 0.**
- **If propagation speed is subluminal, the necessary criterion is**

$$\lim_{|\vec{k}|\to+\infty}\left\{\frac{|\mathrm{Re}\ \omega|}{|\vec{k}|}\leq 1,\ \ |\omega/\vec{k}|\ \text{is bounded}\right\},\ \ \vec{k}\in\mathbb{R}^3$$

- **Dispersion relation of non-relativistic diffusion:** $\omega \propto k^2$ **acausal!**

E. Krotscheck, W. Kundt, Communications in Mathematical Physics 60, 171 (1978)
P. D. Lax, Hyperbolic Partial Differential Equations (2006)

Covariant Stability Implies Causality

- **Criterion for covariant stability**

  Heller, Serantes, Spalinski, Withers, PRL (2023)
  Gavassino, PLB (2023)

$$\mathrm{Im}\ \omega \leq |\mathrm{Im}\ \vec{k}|, \quad \text{for } \vec{k} \in \mathbb{C}^3$$

- **Improved criterion for causality**

  DLW, Pu, PRD 109, L031504 (2024)

$$\mathrm{Im}\ \omega \leq |\mathrm{Im}\ \vec{k}| + b', \ \text{for } \vec{k} \in \mathbb{C}^3$$