EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH
CERN — A&B DEPARTMENT

AB-Note-2007-014 OP

ACCELERATORS FOR PEDESTRIANS

Simon Baird

Abstract

These notes are derived from a course of 11 lectures, which were originally designed as a very basic introduction to circular particle accelerators for people with little or no previous experience in the field. The course itself has since been refined and is now given on a regular basis for the PS accelerator operations technicians. Wherever possible, I have tried to describe the basic theory of synchrotrons and storage rings using simple, easy to follow ideas and models. For this reason, complicated mathematical arguments have been avoided as much as possible. The following topics are covered.

- Matrices and the simple harmonic oscillator
- Relativity, units, Dipole and Quadrupole magnets
- Lattices and Twiss parameters
- Tune shifts, closed orbit, and dispersion
- Chromaticity, Sextupoles, resonances and coupling
- Longitudinal motion and RF
- What is special about electrons and positrons?
- Transfer lines, injection and ejection
- Longitudinal Beam Instabilities
- Transverse Beam Instabilities
- Schottky Noise and Stochastic cooling.

Geneva, Switzerland
February, 2007

Table of contents

Introduction

These notes are derived from a course of 11 lectures, which were originally designed as a very basic introduction to particle accelerators for people with little or no previous experience in the field. The course itself has since been refined and is now given on a regular basis for the PS accelerator operations technicians. Wherever possible, I have tried to describe the basic theory of synchrotrons and storage rings using simple, easy to follow ideas and models. For this reason, complicated mathematical arguments have been avoided as much as possible. Sometimes this approach has meant that there are places where the arguments are not rigorously correct from a mathematical viewpoint. I hope that the experts will excuse these “lapses” as they are only meant to make the basic principles easier to follow for the beginner.

Most aspects of accelerators and storage rings are covered, but since I have not been able to avoid all mathematics the first chapter is a review of the mathematical ideas we will use. For the readers who are at home with matrices, equations of the harmonic oscillator and the notion of phase space, you can skip directly to chapter 2, where things start in earnest.

Chapter 1: Basic notions of the mathematics you will need

A "SCALAR" is simply a number, 2, 3.6 - 7.12, x, y etc. and to transform one scalar into another we just multiply:-

$$X = (n)x \quad \text{or} \quad x = (n^{-1})X$$

A "VECTOR" has two quantities associated with it :-

e.g. x, y- Cartesian co-ordinates r,θ - Polar co-ordinates

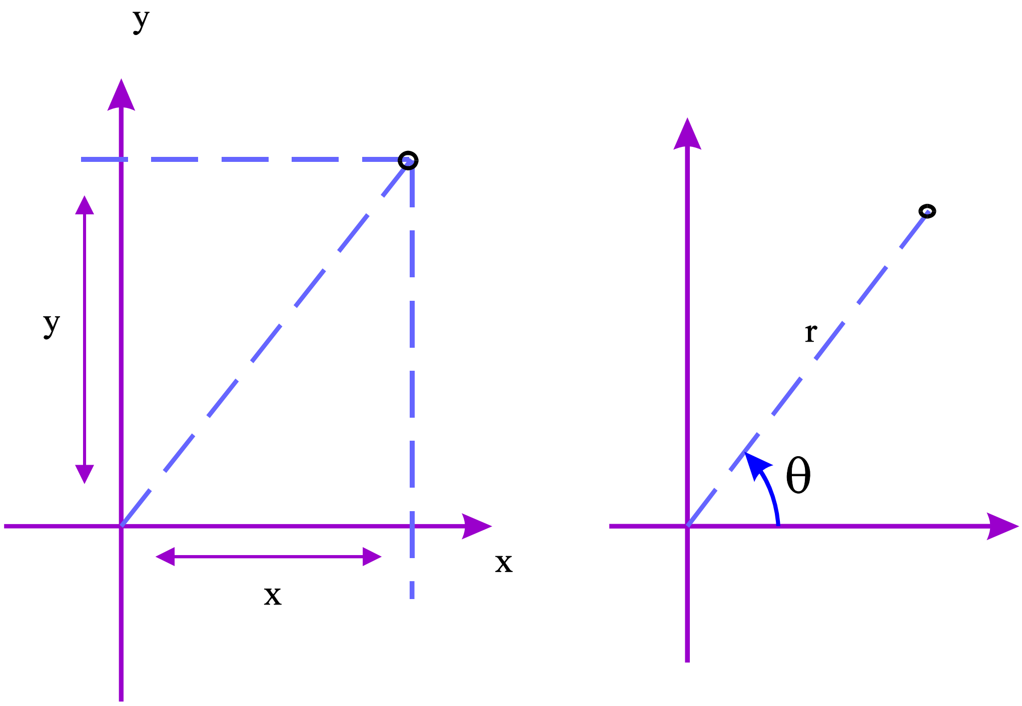


Figure 1: A vector in Cartesian and polar co-ordinates

Anytime we use a vector we need two numbers e.g. (r, θ) or (x, y). In Cartesian co-ordinates the "Length" of the vector, l, is given by:-

$$l = \sqrt{x^2 + y^2}$$

If we multiply a vector A (x, y) by a Scalar, N, the result is another vector, NA = (Nx, Ny), i.e. this changes only the length, but not the direction of the vector A. (see figure 2). To get independent control of both length and direction i.e. to move from point A, in figure 2, to any other point B we need two equations:-

$$x_{new} = ax_{old} + by_{old}$$
$$y_{new} = cx_{old} + dy_{old}$$

This is rather clumsy however.

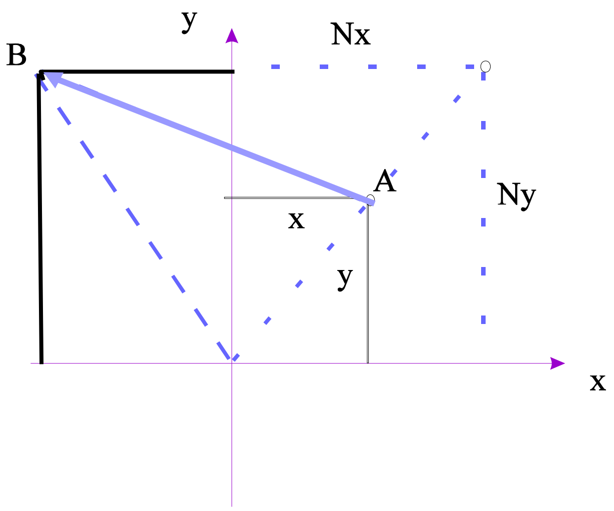


Figure 2: Vector transformation

Therefore, these two equations we write as a single equation:-

$$\overline{B} = M\overline{A}$$

or

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

M is a matrix with two rows and two columns, and therefore 4 elements (a, b, c, d). A and B are matrix representations of vectors shown in Figure 2. They have 2 rows and 1 column, which means they have 2 elements, which are their x and y co-ordinates.

A matrix can have any number of columns and rows, but our matrices will normally only have 2 rows and 2 columns, called “2x2” matrices, or 2 rows and 1 column, called “2x1” matrices.

Matrix Multiplication

We need to get from the matrix equation back to our original equations.

$$x_{new} = ax_{old} + by_{old}$$
$$y_{new} = cx_{old} + dy_{old} \quad (1)$$

These two equations must be the same

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \quad (2)$$

Therefore equations (1) and (2) above define the rules for matrix multiplication. We can also multiply two matrices, but we must be careful of the order of the matrices, since in matrix multiplication A.B ≠ B.A

$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

In this case the individual elements of the final matrix are given by:

$i = ae + bg, \; j = af + bh, \; k = ce + dg, \; l = cf + dh$

But why bother to multiply matrices ?

In figure 3, matrix M1 will transform from position 1) to 2)

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

matrix M2 will transform from position 2) to 3)

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M2M1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Now, if we define M3 = M2.M1 then this matrix will transform directly from position 1) to 3)

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M3\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

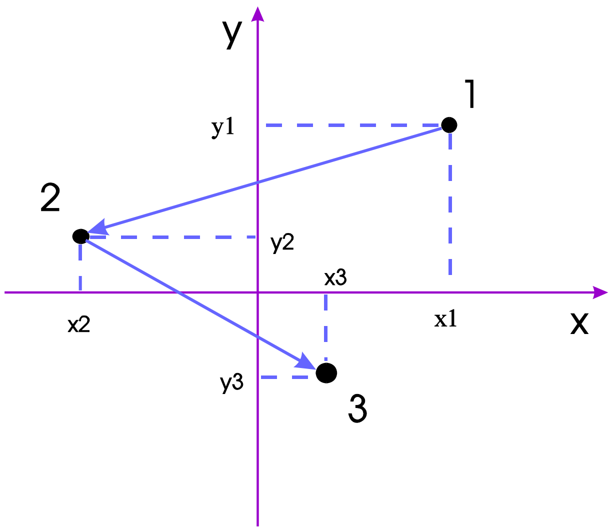


Figure 3: Transformation of p1 to p3 via p2

In our accelerator we will use matrices to describe the effects of the various magnetic elements on our particles. Instead of x and y co-ordinates, we will use the position and angle of each particle. Now if we know the position and angle of the particle at one point, and we want to calculate its position and angle at another point we simply need to multiply all the matrices together between the two points to give us a single 2x2 matrix. Then multiplying our original position and angle by this 2x2 matrix will tell us the particle's position and angle at the second point. Once we have done the matrix multiplication for the first calculation, we can recalculate the final position and angle very quickly for any other initial particle co-ordinates. This is a very powerful method for calculating.

Some special cases

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \quad \begin{matrix} x_2 = x_1 \\ y_2 = y_1 \end{matrix}$$

^

This is called a Unit matrix, it has no effect

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & o \\ o & a \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \quad \begin{matrix} x_2 = ax_1 \\ y_2 = ay_1 \end{matrix}$$

^

This is called a Diagonal matrix, it does not mix x and y

Our matrices have only two "rows" and two "columns", but remember that a matrix can have any number of rows and columns, but we will only use "2x2" matrices.

We can now transform the vector $\begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$ to vector $\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix}$

Remember $\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$ or $\overline{B} = M\overline{A}$

In order to make the reverse transformation we need another matrix (M^{-1})

$\overline{A} = (M^{-1})\overline{B}$ or $\overline{B} = MM^{-1}\overline{B}$ Remember the order is important.

(M^{-1}) is the "inverse" or "reciprocal" matrix of (M). For "scalar" numbers $MM^{-1} = 1$ = Unit matrix, which has no effect.

Inverse of a 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The term [ad - bc] is called a determinate and it is just a number

<u>A useful example</u>

Changing the current in the QF , QD quadrupoles in the accelerator changes the horizontal and vertical tune values (Qx, Qy), however each quadrupole has a effect on both Qx and Qy.

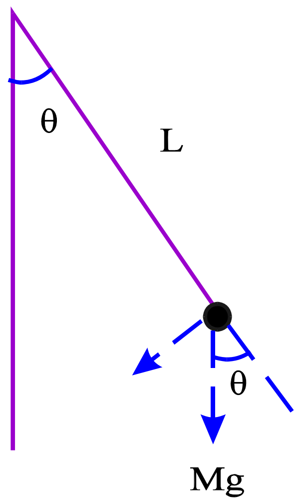
If ΔI = the current change in quadrupole, then we can write this as a matrix equation:-

$$\begin{pmatrix} \Delta Qx \\ \Delta Qy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix} \quad \text{or} \quad \Delta Q = M\Delta I$$

Now if we perform an experiment and change If and Id by ΔIf and ΔId and we can measure the resulting change ΔQx, ΔQy. Now we can calculate M and then M^{-1}. Now we can use the matrix equation $\Delta I = M^{-1}\Delta Q$, which is just the inverse of our original equation, to calculate the currents, ΔIf and ΔId, which are required to give any change in Qx, Qy.

<u>What is a "differential equation" ?</u>

<u>A Pendulum!</u> (for example)



The sign of the restoring force depends on displacement. We will try to find an equation to describe this motion:-

a) Distance of weight from centre = $L\Theta$ (Θ is small)

b) Velocity of weight = change of distance/time = $\frac{d(L\Theta)}{dt}$

c) Acceleration of weight = $\frac{d^2(L\Theta)}{dt^2}$

But force = mass x acceleration (remember the force opposes the motion)

$$mg\sin\Theta = m\left(-\frac{d^2(L\Theta)}{dt^2}\right) \quad \text{(L = constant)}$$

$$\frac{d^2\Theta}{dt^2} + \left(\frac{g}{L}\right)\sin\Theta = 0$$ This is a differential equation.

If Θ is small then $\sin\Theta \rightarrow \Theta$

Therefore $\frac{d^2\Theta}{dt^2} + \left(\frac{g}{L}\right)\Theta = 0$

We would expect the answer to be an oscillation at some frequency "ω", since common sense tells us that the weight will swing backwards and forwards, but what is ω?

Guess a solution $\Theta = A\cos(\omega t)$ where A = Amplitude of the oscillation.

Therefore $\frac{d\Theta}{dt} = -A\omega\sin(\omega t)$

and $\frac{d^2\Theta}{dt^2} = -\omega^2 A\cos(\omega t)$

If we put this back into the original equation (2) then:-

$$-\omega^2 A\cos(\omega t) + \frac{g}{L} A\cos(\omega t) = 0$$

Comparing this with the original equation (2)

$$\omega^2 = \frac{g}{L} \qquad \omega = \sqrt{\frac{g}{L}}$$

This gives us the final equation for the motion of the pendulum as:-

$$\Theta = A\cos\sqrt{\frac{g}{L}}\,t$$

This style of equation:- $$\frac{d^2x}{dt^2} + kx = 0$$

Is the sort of equation we will use to describe the particles as they move around our accelerator. This sort of equation describes oscillatory motion. We may sometimes need to add an extra term to describe a drag, which slows down the weight as it moves. For example imagine the same pendulum immersed in a bath of oil. Now there is an extra force on the pendulum, the drag of the oil, which will tend to reduce the oscillation amplitude. It would seem reasonable to say that this extra force will get stronger the faster we try to move the pendulum.

i.e. “drag” $\propto$ velocity of the weight.

Now our original equation becomes:-

$$\frac{d^2\Theta}{dt^2} + 2\alpha\frac{d\Theta}{dt} + \omega^2\Theta = 0 \qquad \omega = \sqrt{\frac{g}{L}}$$

This is a “damped” harmonic oscillator.

Solution $\Theta = Ae^{-\alpha t}\cos(\omega t)$

The exponential factor reduces the amplitude as a function of time. This is very important in electron machines, where particles “lose” energy over time due to synchrotron radiation emission.
You can verify the above solution exactly as before except the differentiation is more complicated.

Remember. $$\frac{de^{-xt}}{dt} = -xe^{-xt}$$

But it does work !

Back to our simple oscillator

$$\frac{d^2x}{dt^2} + w^2x = 0$$ Use *x* instead of Θ $\left(w^2 = \frac{g}{L}\right)$

For any system where the restoring force depends on displacement the solution will be $x = x_0.\cos(wt)$. Therefore the velocity of pendulum, particle or whatever will be given by

$$\frac{dx}{dt} = -wx_0.\sin(wt)$$

If we plot the velocity .v. displacement we will get an ellipse.

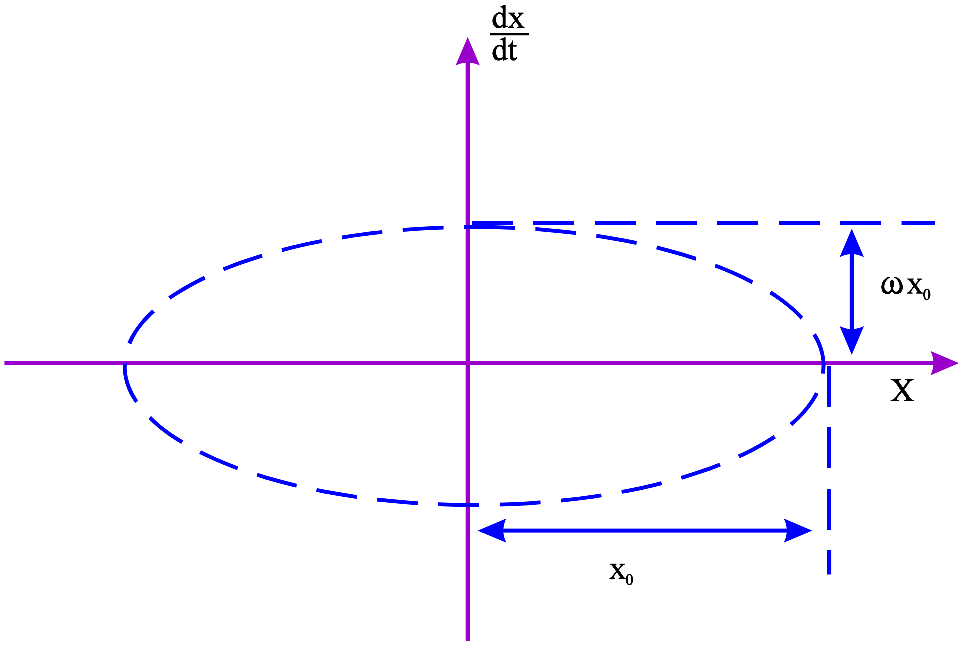


Figure 4: Phase space ellipse for a pendulum

As we go from (wt) to $(wt + 2\pi)$, this ellipse repeats itself.
This continues for $(w + 2\pi)(w + 4\pi)$ etc......

$\Phi =$ wt is called the phase angle, and our ellipse is a “phase diagram” or “phase space plot”.

This *x* axis will normally be either displacement in <u>position</u> or time.
the *y* axis will be either <u>angle</u> , energy or velocity
N.B. This is the same position and angle as we saw for the matrices !!

A particle in the accelerator or the weight on the end of the pendulum will move around this plot. $\Phi \to \Phi + 2\pi$, with displacement and velocity given by:-

$$x = x.\cos\Phi$$

$$\frac{dx}{d\Phi} = -x.\sin\Phi$$

Exercises 1

1) Find the products of the following matrices.

a) $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & l_2 \\ 0 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$

2) The matrix relating "Q" of a machine to quadrupole currents is :-

$$\begin{pmatrix} \Delta qx \\ \Delta qy \end{pmatrix} = \begin{pmatrix} 1.2 & 0.3 \\ 0.2 & 2.1 \end{pmatrix}\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix} = m\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix}$$

a.) What is the "reciprocal" or "inverse" of m (i.e. m^{-1}) ?

b.) What values of $\Delta If, \Delta Id$ are needed to change only ΔQx by 0.1 ?

3) You can measure Q*x* and Q*y* in your accelerator. Suggest the measurements necessary to evaluate the matrix 'm' in question (2)

4) A mass 'm' is hanging on a spring, the weight is pulled down a distance x and released, the restoring force of the spring per unit displacement is 'k', what is the frequency of oscillation? Does the frequency depend upon the initial amplitude?

5) Draw a phase plot of the motion of the weight in, 4) by plotting displacement .v. velocity.

As you increase the "phase angle" Φ, do you travel clockwise or anti clockwise around the ellipse?

Solutions 1

1) a. $\begin{pmatrix} 14 \\ 6 \end{pmatrix}$ d. $\begin{pmatrix} 1 & l_1 + l_2 \\ 0 & 1 \end{pmatrix}$

b. $\begin{pmatrix} mx \\ my \end{pmatrix}$ e. $\begin{pmatrix} 1 & l \\ \frac{-1}{f} & 1 - \frac{l}{f} \end{pmatrix}$

c. $\begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$

2) $ad - bc = 2.46$

Inverse of matrix = $\frac{1}{2.46}\begin{pmatrix} 2.1 & -0.3 \\ -0.2 & 1.2 \end{pmatrix}$

$$\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix} = \begin{pmatrix} 0.85 & -0.12 \\ -0.08 & 0.49 \end{pmatrix}\begin{pmatrix} \Delta Qx \\ \Delta Qy \end{pmatrix}$$

$$= \begin{pmatrix} 0.85 & -0.12 \\ -0.08 & 0.49 \end{pmatrix}\begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$$

$\Delta If = 0.085$

$\Delta Id = -0.008$

3) Change If by ΔI and leave Id fixed, then measure the changes $\Delta Qx \Delta Qy$

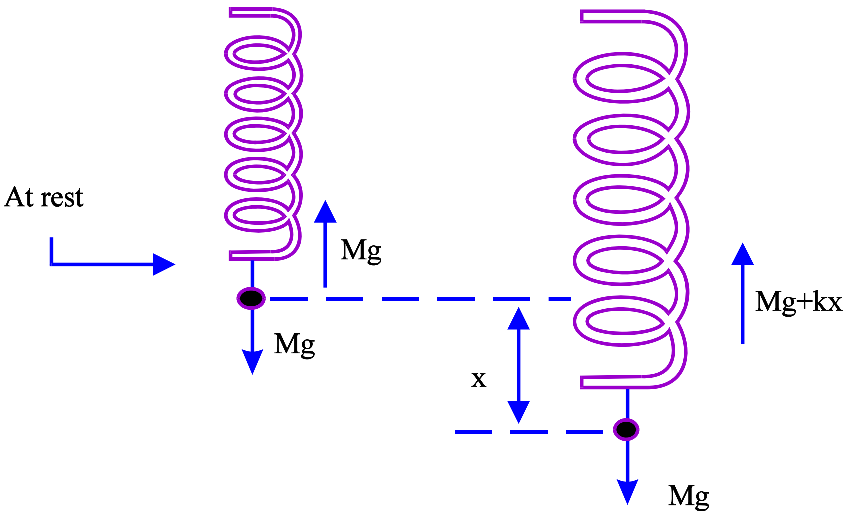
now $\begin{pmatrix} \Delta Qx \\ \Delta Qy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \Delta If \\ \Delta Id \end{pmatrix}$ But $\Delta Id = 0$

$\therefore\ a = \frac{\Delta Qx}{\Delta If}$ and $c = \frac{\Delta Qy}{\Delta If}$

similarly for Id leave If fixed.

$\therefore\ b = \frac{\Delta Qx}{\Delta Id}$ and $d = \frac{\Delta Qy}{\Delta Id}$

4)



Resulting force = Kx But F = Ma ...... Newton again.

$$Kx = -m\frac{d^2x}{dt^2}$$

There is a negative sign because the acceleration always opposes the direction of motion
Therefore:

$$\frac{mdx^2}{dt^2} + Kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{K}{m}x = 0$$

exactly as for the pendulum.

$$x = x_\circ \cos(\omega t + \Phi)$$
$$\frac{dx}{dt} = -x_\circ\omega \sin(\omega t + \Phi)$$
$$\frac{d^2x}{dt^2} = -x_\circ\omega^2 \cos(\omega t + \Phi)$$
$$\therefore \omega^2 = \frac{K}{m} \qquad \omega = \sqrt{\frac{K}{m}}$$

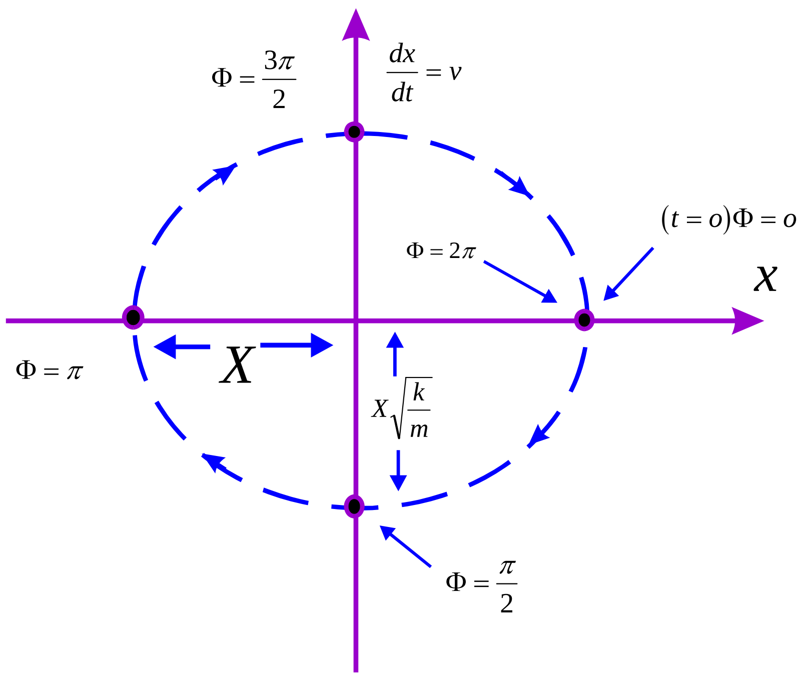
But when t = 0, x = X, therefore x_0 = X and $\varphi = 0$

Solution is $x = X\cos\sqrt{\frac{K}{m}}t$

Frequency does not depend on the amplitude x

5) $x = X\cos\sqrt{\frac{K}{m}}t$

$$v = \frac{dx}{dt} = -x\sqrt{\frac{K}{m}}\sin\sqrt{\frac{K}{m}}\,t$$



Travel clockwise around plot.

Chapter 2: Relativity, units, dipoles and quadrupoles

This is the chapter in which we really start to look at an accelerator

Relativity

As we accelerate our particles from rest, at first the speed increases, but as the energy gets higher and higher, this increase in speed becomes smaller and smaller as we approach the velocity of light. In this case increasing the particle's energy only serves to increase it's mass as we can no longer accelerate past the velocity of light. See figure 1. The low energy part of figure 1 is the everyday world we see, where velocities are much less than the velocity of light and Newtonian mechanics describes motion. The high energy part of figure 1 is the world of the high energy particle, where velocities are close to the velocity of light and Relativity is needed to describe motion.

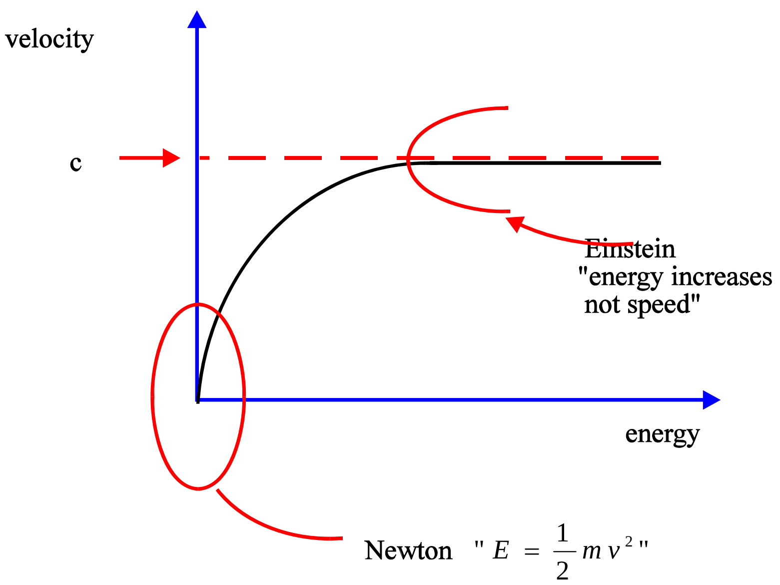


Figure 1: A particle's velocity as a function of increasing energy

The following are definitions, which define the energy of any moving particle:-

Energy of a particle at rest = $E_0 = m_0c^2$

Energy of a moving particle = $E = \gamma E_0 = \gamma m_0 c^2$ (Definition of γ)

$\beta = \frac{v}{c}$ (definition of β)

For highly relativistic particles γ is large i.e. total energy>> rest energy, and $\beta \Rightarrow 1$

Einstein's Relativity gives $\rightarrow E = mc^2$, and from the definition of β:-

$$\beta = \frac{mv}{mc} = \frac{mvc}{mc^2}$$

But it is always true that a particle's momentum is given by:-

$$p = mv$$

Therefore from the above two equations $\beta = \frac{pc}{E}$, but for highly relativistic particles $\beta \approx 1$, therefore $p = \frac{E}{c}$. This is true provided that the energy of the particle is large compared with its rest energy i.e. Υ is large. This means that for high energy particles the total energy and the momentum are numerically virtually equal. But what are the units that will be used to describe the particle's energy and momentum?

Units

The unit of energy used for particle energies is the electron volt (eV), and it is defined as: 1 eV is the energy acquired by an electron in a potential of 1 Volt.

1 KeV = 10^3 eV; 1 MeV = 10^6 eV; 1 GeV = 10^9 eV etc. etc. etc.

We saw previously that momentum is given by $p = \frac{E\beta}{c}$, therefore the units of momentum are eV/c. At high energies (more exactly for highly relativistic particles), energy and momentum are numerically equal (or very close), but be careful this is not true at low energies.

How do we define low energies?

Remember $E = E_0\gamma$ or $m_0c^2\gamma$, therefore, at rest when $\gamma = 1$, $E_0 = m_0c^2$. This allows us to express the rest mass of a particle (it's mass when it is not moving) in energy units.

For a proton or an antiproton E_0 = 938 MeV

In the PS for protons at 26 GeV/c, $\gamma = 28.7$ $\beta = 0.999$ therefore "energy = momentum"

In the AAC for antiprotons at 3.5 GeV/c, $\gamma = 4.7$ $\beta = 0.977$ and again we can say that "energy = momentum"

In LEAR for antiprotons at 0.6 GeV/c, γ = 1.19 β = 0.54 therefore the energy = 1.12 GeV and the momentum = 609 MeV/c. Therefore energy and momentum are far from "equal".

For an electron E_0 = 0.51 MeV or M_0 = 0.51 MeV/c^2, therefore even at a momentum of only 120 MeV/c, γ = 240 β = 0.9999 and the electron energy and momentum are numerically equal.

Co-ordinate system

We will need to describe particles as they move around the ring, therefore we will use a 3-D coordinate system, where X is the transverse horizontal axis, Y is the vertical transverse axis and s or θ is the longitudinal axis, see figure 2.

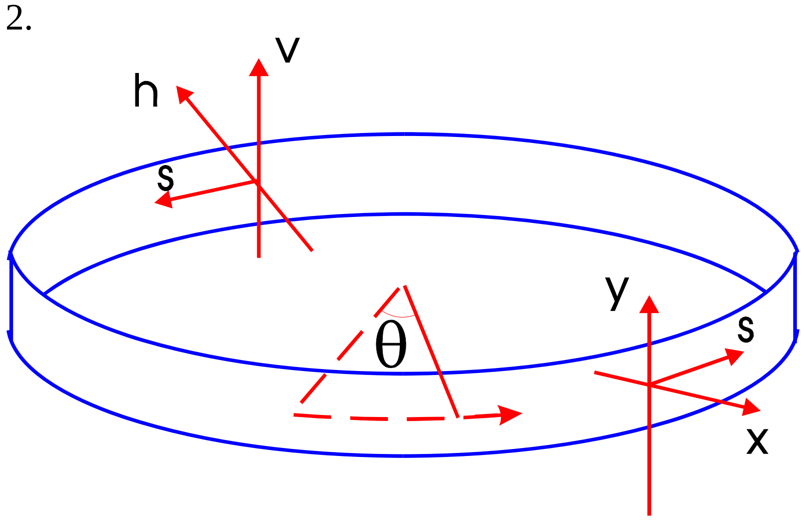


Figure 2: Coordinate system for particles in an accelerator

y = vertical
x = horizontal
s = longitudinal ($\Theta \rightarrow 0 \text{to} 2\pi$ is called the "longitudinal phase")

Now finally we are in a position to look at a real storage. We will need a magnetic field to contain the particles on a particular radius orbit. This radius will be defined by the momentum of the particle and the strength of the magnetic field. It will be a **Dipole field.** See Figure 3.

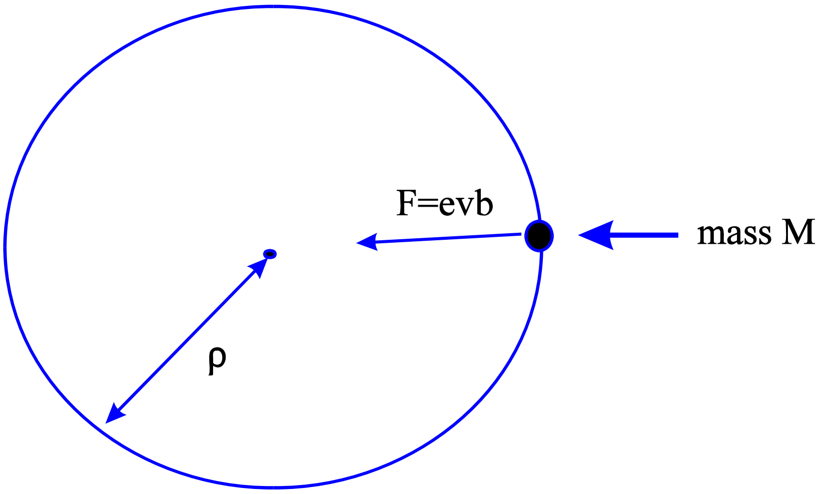


Figure 3: Motion of a particle in a dipole magnetic field, the field is in/out of the page

The force, **evB,** on charged particle in a magnetic dipole field of strength **B,** moving with a velocity **v**, will be equal to "mass x acceleration" towards centre of it's circular path:-

$$F = \frac{mv^2}{\rho}, \text{ where } \rho = \text{radius of curvature of the path}$$

$$F = evB = \frac{mv^2}{\rho}$$

(remember p = momentum = mv)

$$B\rho = \frac{mv}{e} = \frac{p}{e}$$

(Bρ) is called the "magnetic rigidity", and if we put in all the correct units we get :-

(Bρ) = 33.356p (KG.m) = 3.3356p (T.M) (if p is in "GeV/c")

Obviously we would now like to accelerate beam in our accelerator or storage ring, but ρ is fixed by the size and/or layout of the magnets. This means that as we increase the particle momentum, p, we must increase the magnetic field. In reality we use a series of dipole magnets to bend the beam around a closed path (or orbit). Therefore it is interesting to examine what happens to a particle in a dipole magnet.

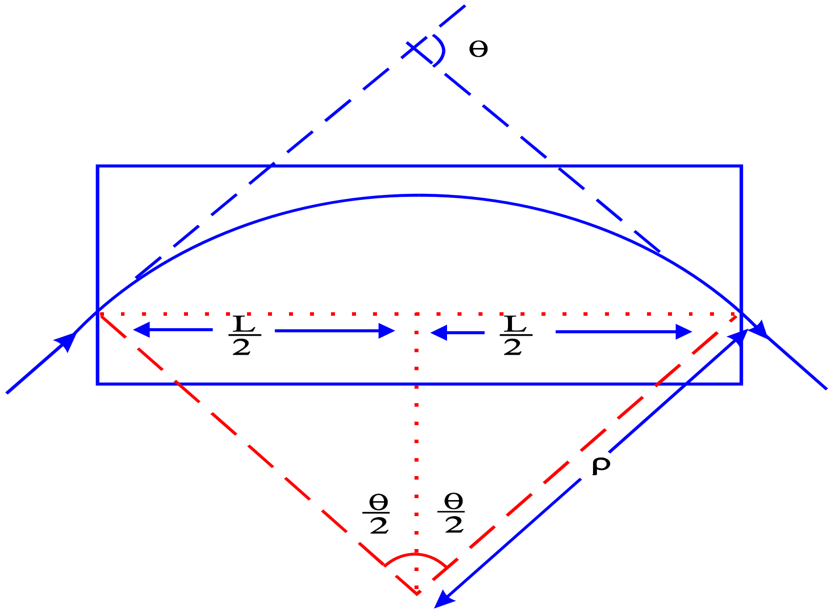


Figure 4: Effect of a uniform dipole field of length L and field B

The beam is deviated by an angle θ, where $\sin\frac{\Theta}{2} = \frac{L}{2\rho} = \frac{1}{2}\frac{LB}{(B\rho)}$

If θ is small then sin θ ≅ θ then:-

$$\Theta = \frac{LB}{(B\rho)} \qquad (B\rho) = \text{“magnetic rigidity”} \propto \text{momentum.}$$

In accelerators we often see magnets called Sector magnets, which are curved so that the beam enters and leaves magnet at 90 degrees to pole faces. See Figure 5.

This is great for one particle but a useful accelerator or storage ring must contain more than 1 particle! So let's add a second particle to the first one in our machine. (All particles have the same momentum) Let us consider first only the horizontal (x) plane. Figure 6 shows the trajectories for two particles in a dipole field, each with a different initial angle.

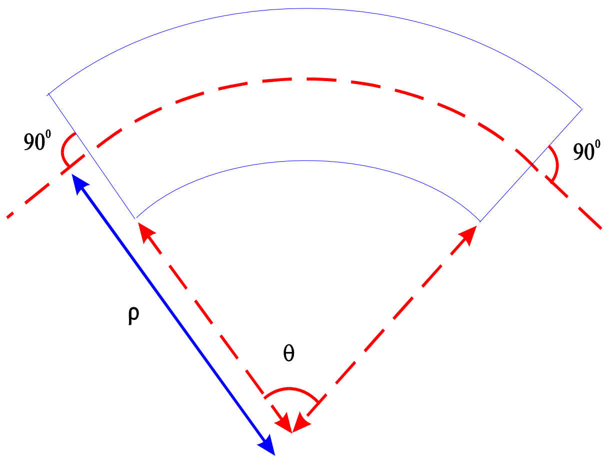


Figure 5: A sector magnet

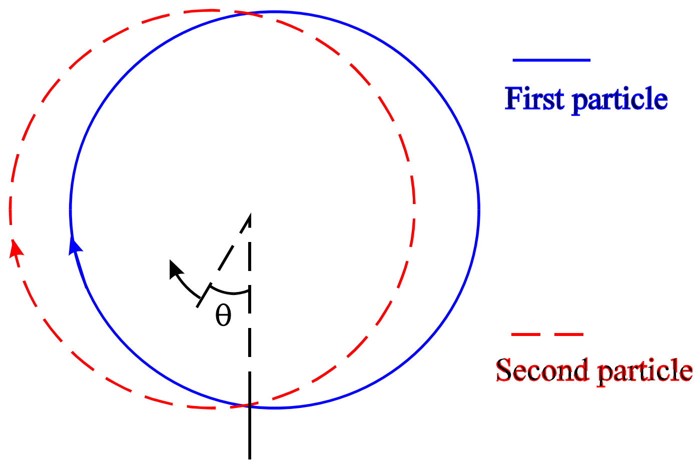


Figure 6: Two particles in a dipole field, with different initial angles

Now if we unfold figure 6 into a "straight line", and plot the horizontal displacement of the first particle with respect to the second, we see that the second particle oscillates around the first particle. See figure 7. This is a **Betatron Oscillation**. This oscillation forms the basis of all transverse motion in an accelerator.

So far we have only considered the horizontal plane. If we now allow a difference in the initial vertical angle, we get a spiral that never closes. Therefore although particle motion in our simple dipole magnet is "stable" i.e. the trajectories close, in the horizontal plane, it is "unstable" i.e. different trajectories do not close in the vertical plane. So we need some extra focusing, in addition to our dipoles, which will force particles back towards the "ideal" or "reference" trajectory. This extra focusing we get with a **Quadrupole magnet.**

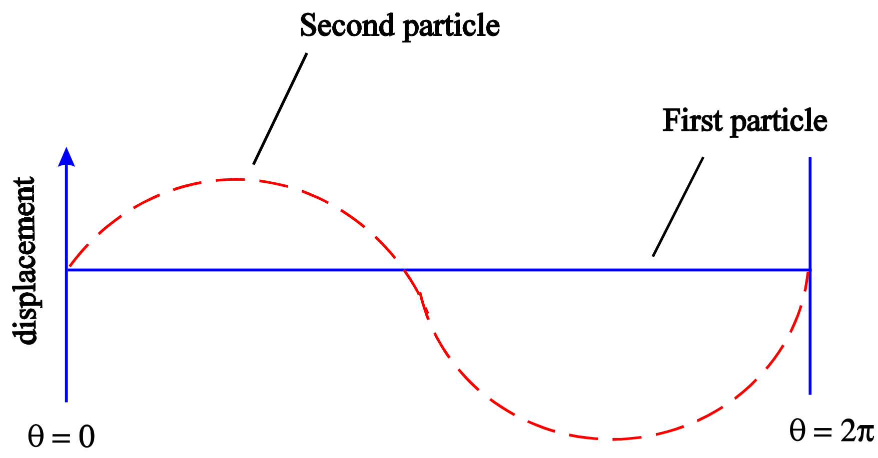


Figure 7: Transverse oscillation of the second particle around the first in our "two particle" storage ring from figure 6.

Effect of a quadrupole

As it's name suggests a quadrupole has 4 poles, (2 North and 2 South) arranged symmetrically around the beam, see figure 8.

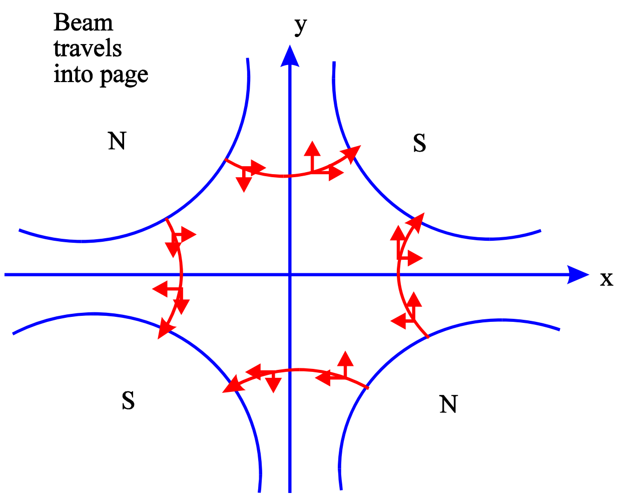


Figure 8: Magnetic field in a quadrupole

There is no field along the central axis.

As we move out along the x (horizontal) axis the field is vertical and is given by:-

$By \propto x$

Along the y (vertical) axis the field is horizontal and is given by:-

$Bx \propto y$

We characterise a quadrupole by a field gradient, $\frac{d(By)}{dx}$ which is constant and is called K, the "gradient" of the quadrupole. The units for K are $KGM^{-1}(TM^{-1})$. We may also see another definition $k = \frac{K}{(B\rho)}(m^{-2})$, where k is the "normalised" gradient.

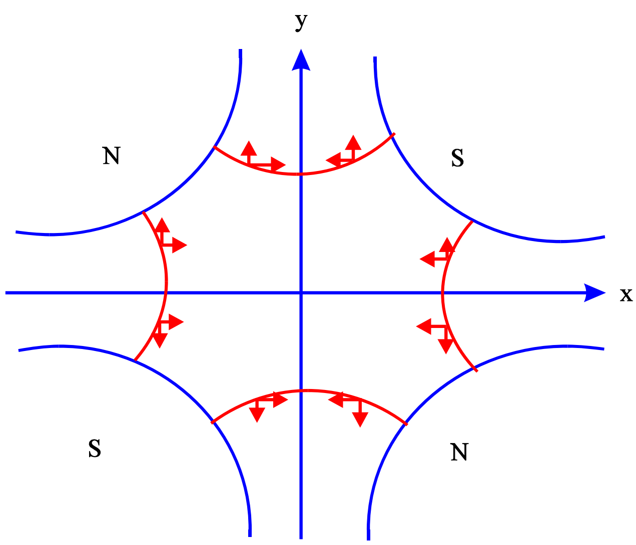


Figure 9: Force on a particle moving through a quadrupole

Therefore a particle which deviates from the central axis of the quadrupole in the horizontal plane, is deflected back towards the centre of the magnet. Thus this magnet focuses particles in the horizontal plane. Unfortunately the opposite is true in the vertical plane, and it defocuses the beam vertically!
This example of a Quadrupole is a Focusing Quad. (QF). It focuses the beam in the horizontal plane and defocuses the beam in the vertical plane. If we rotate the poles by 90 degrees it would become a Defocusing Quad. (QD). i.e. focus in the vertical plane and defocus in the horizontal plane.

We can learn a lot about the behaviour of the beam by considering these quadrupoles as “thin lenses” and the beam as a diverging light beam.
Imagine a series of lenses (we need to do this twice, once in the horizontal plane and once in the vertical plane)

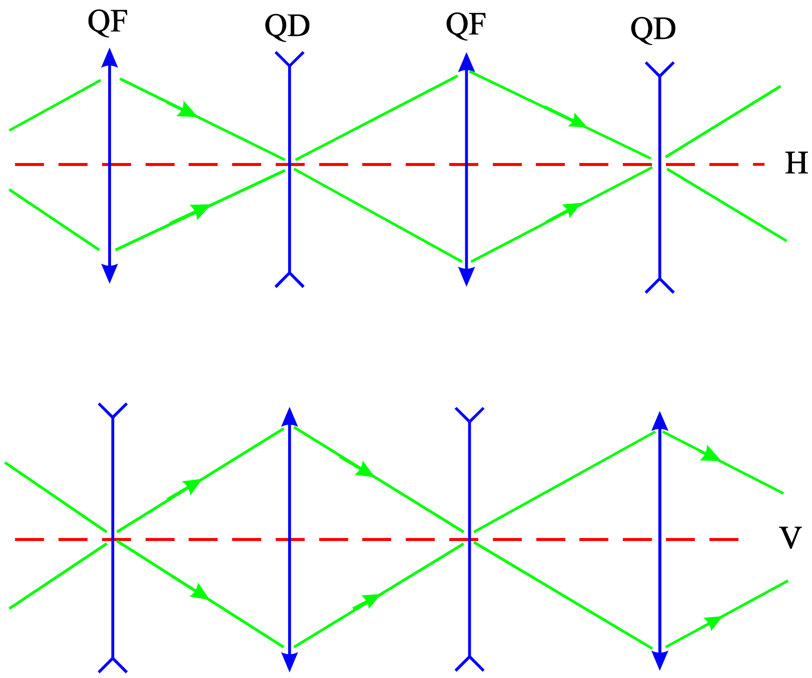


Figure 10: Light rays passing through a series of focusing and defocusing lenses

In Figure 10, the lenses, which are concave in one plane, are convex in the other. In both cases the concave lenses will have little effect as the light passes very close to their centre, and the net result is that the light rays are focused in both planes.
Our accelerator consists of a series of dipoles to constrain the beam to follow some closed path, often approximately circular, focusing and defocusing quadrupoles to provide the horizontal and vertical focusing needed to contain the beam in the transverse (X and Y) directions. A very common combination of focusing and defocusing sections is the FODO lattice. This consists of alternate focusing and defocusing sections with non focusing - drift sections between them. See figure 11.

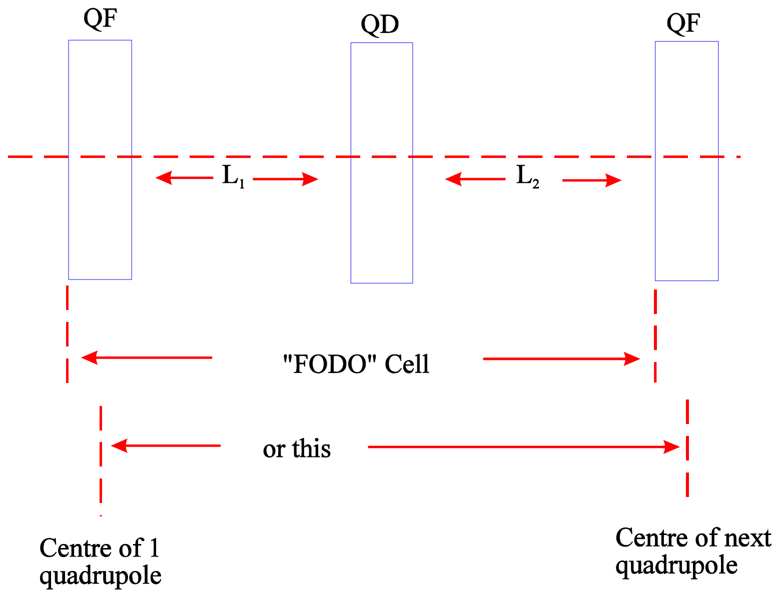


Figure 11: Layout of a FODO Cell

As the particles move around the accelerator or storage ring, whenever their divergence (angle) causes them to stray too far from the central trajectory the quadrupoles focus then back towards the central trajectory. This is rather like a ball rolling around a circular gutter. See Figure 12.

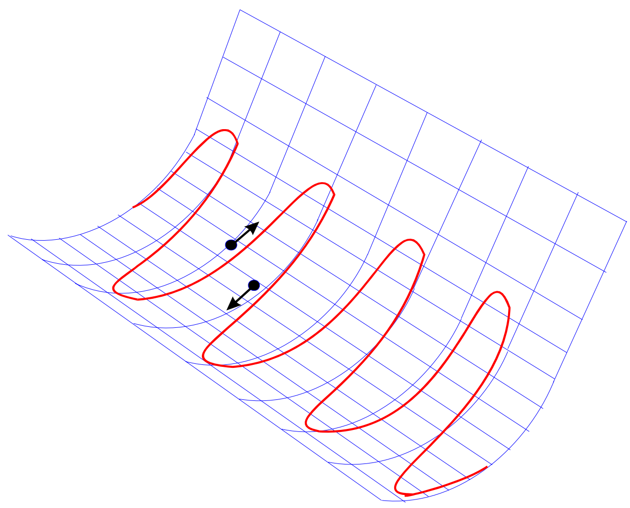


Figure 12: A ball rolling along a gutter

We characterise the position of the ball (particle) in this transverse motion by two things: Position or displacement from central path, and angle with respect to central path. See Figure 13.

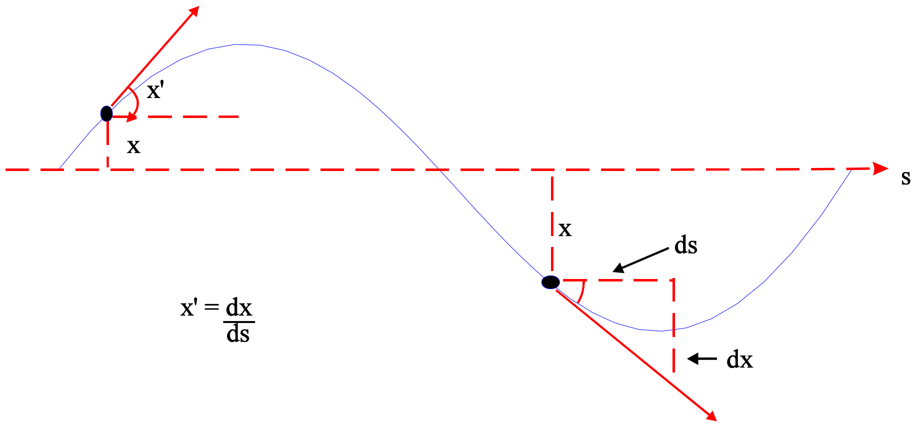


Figure 13: Transverse position and angle displacement for a particle moving around our accelerator.

These transverse oscillations are called **Betatron Oscillations**, and they exist in both horizontal and vertical planes. The number of such oscillations/turn is Qx or Qy. Until now we thought of these transverse oscillations as varying in time.

$$(\mathrm{SHM})\frac{d^2x}{dt^2} + Kx = 0 \rightarrow x = A\cos(\omega t)$$

But since "s" increases linearly with time it is also possible to express this equation in terms of s, the distance around the accelerator ring.

$$x = A\cos\left(\frac{\omega}{v}s\right) \qquad \text{Remember } s = vt\ ;\ \ v = \text{velocity around the ring}$$

If Qx or Qy is non-integer, then the phase of the betatron oscillator will be different at each turn. Now we can set up a "detector" at one point around our ring and measure the transverse position and angle on many consecutive turns. The transverse motion will be simple harmonic motion and will obey the equation.

$$\frac{d^2x}{dt^2} + Kx = 0$$

K is the restoring or focusing force due to the quadrupoles. Remember x = displacement from central orbit and x' = angle with respect to the central orbit.

$$x = A\cos(\omega t + \phi)$$
$$x' = -A\omega\sin(\omega t + \phi)$$

ϕ is just a constant phase factor, which tells us about the initial phase of the oscillation at t = 0.
Now with many x , x' "pairs" measured over many turns we can make a phase space plot. x . v . x'. See figure 14:

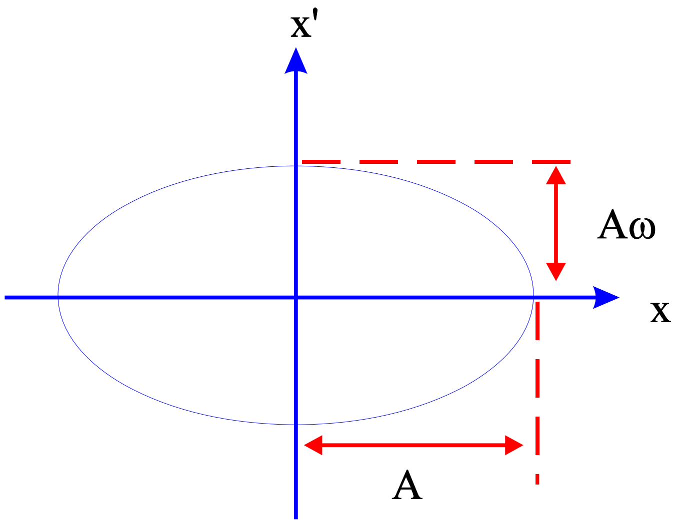


Figure 14: Phase space ellipse for a single particle

In reality our beam will consist of many particles each with different amplitudes A for their own betatron oscillations and different phases ϕ, so on any single turn we will see (if we could distinguish and measure each individual particle) many points on the above plot each corresponding to a particular particle. We will come back to this point in the next chapter.

In fact it is more usual to describe the transverse motion of particles as a function of s, the distance around the accelerator.

$$x = A\cos\left(\frac{\omega}{v}s + \phi\right)$$

$$x' = -A\frac{\omega}{v}\sin\left(\frac{\omega}{v}s + \phi\right)$$

Again we can represent the particles position on a phase space plot. However we can also represent the particles transverse position and angle by a column matrix $\begin{pmatrix} x \\ x' \end{pmatrix}$. As we move around the ring $\begin{pmatrix} x \\ x' \end{pmatrix}$ will vary under the influence of the dipoles, quadrupoles and empty (drift) spaces. We express this modification in terms of a TRANSPORT MATRIX (M). If we know x_1 and x_1' at some point s_1, for a point s_2 after the next element in the accelerator ring:-

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = M\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

If we split our machine into elements Dipoles, Focusing and Defocusing quadrupoles, and drift spaces and we can devise matrices for each of these, then multiplying them all together will enable us to see exactly what happens to a particle as it moves around the machine.

1) Drift space - No magnetic fields, length = L

$$x_2 = x_1 + Lx_1'$$

$$x_2' = 0 + x_1'$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

Same in both planes Horizontal and Vertical.

2) Quadrupole of length, L

Horizontal plane - remember $By \propto x$ or $By = Kx$

Vertical plane - remember $Bx \propto y$ or $Bx = Ky$

The deviation is given by $\dfrac{LBy}{(B\rho)}$

Therefore we can rewrite the new position and angle as:-

$$x_2 = x_1 + 0$$

$$x_2' = \frac{LBy}{(B\rho)} + x_1'$$

But $\quad By = Kx$

$$\therefore x_2 = x_1 + 0$$

$$\therefore x_2' = \frac{LK}{(B\rho)} x_1 + x_1'$$

$$\therefore \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{LK}{(B\rho)} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

Here we must be careful with signs. If we define f = focal length = $\frac{(B\rho)}{KL}$ and f is positive for focusing effect, then:-

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

For a horizontal focusing quadrupole (QF)

f (focal length) is positive in horizontal plane

f (focal length) is negative in vertical plane

For a vertically focusing quadrupole (QD)

f (focal length) is negative in horizontal plane

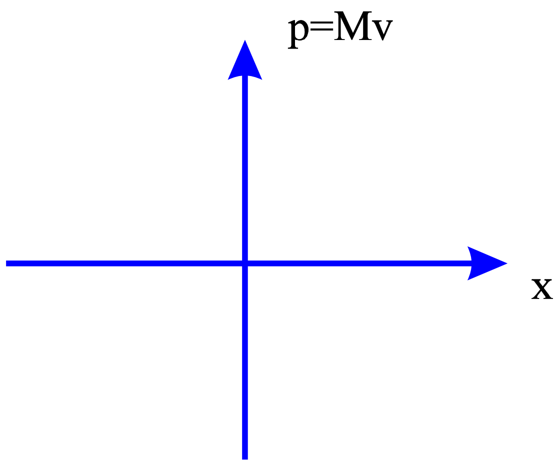
f (focal length) is positive in vertical plane.

3) Dipole - This is essentially just a drift space as a dipole bends the central particle and all others by the same amount. In the vertical plane a horizontal dipole has no effect, provided of course it is correctly aligned! Horizontally, however, there is a small focusing effect if path lengths or trajectories are different inside the magnet. Remember, at the beginning of this section, we looked at a particle moving in a dipole field, and saw that particles with different initial angles were indeed focused. However, for our purposes we will treat a dipole as a simple drift space.

For transverse motion we can construct transport matrices corresponding to drift spaces and quadrupoles, which will describe how the particles move around the machine. We will use the matrices but at the same time we will have to put together these matrices, which describe real discrete focusing of our quadrupoles, with our “SHM” type oscillator picture of the betatron oscillators where the restoring force is constant all round the ring. This is rather like saying what happens to the motion of a ball rolling around a gutter we allow the shape of the gutter to vary as we move along it?

<u>**Exercises 2**</u>

1) A pendulum starts at maximum amplitude A at time, t=0, what is the path traced out by it's motion in the following phase diagram:



2) A quadrupole doublet consists of two thin lenses (focal lengths f1 and f2), separated by a length *l*. Show that the product matrix describing the complete system is:-

$$\begin{pmatrix} 1-\frac{\ell}{f_1} & \ell \\ \frac{-1}{f^*} & 1-\frac{\ell}{f_2} \end{pmatrix} \qquad \text{where} \quad \frac{1}{f^*} = \frac{1}{f_1}+\frac{1}{f_2}-\frac{\ell}{f_1 f_2}$$

A FODO cell may be considered as, one such matrix with f_1 = 2f and f_2 = -2f, followed by another cell with f_1 = -2f, f_2 = 2f. Write down these two matrices from the middle of one QF to the middle of the next, and show that this FODO cell, from the middle of one QF to the middle of the next, can be represented by the matrix :-

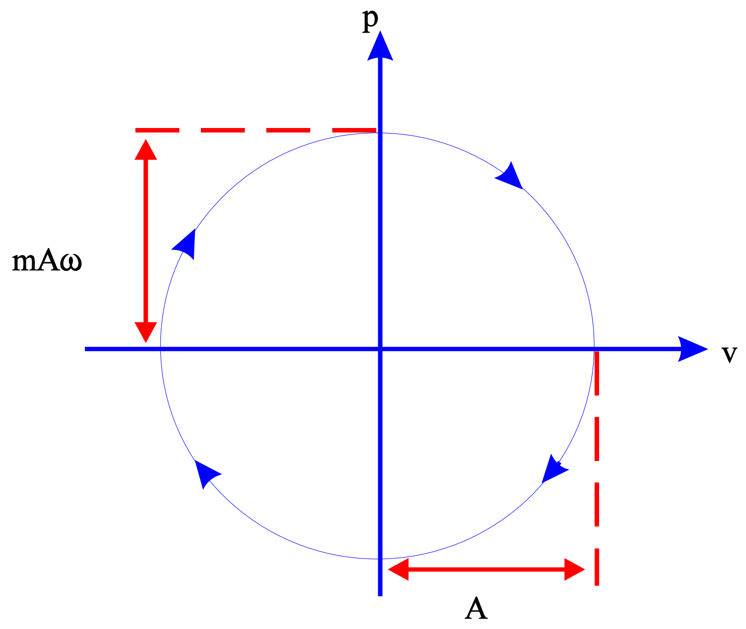
$$\begin{pmatrix} 1-\frac{\ell^2}{2f^2} & 2\ell\left(1+\frac{\ell}{2f}\right) \\ \frac{-\ell}{2f^2}\left(1-\frac{\ell}{2f}\right) & 1-\frac{\ell^2}{2f^2} \end{pmatrix}$$

3) A ball rolls down a gutter of radius R. It's mass is M. It's velocity is v.
a) What is the transverse equation of motion (use time as the variable)
b) What is the equation of motion along the gutter (remember S = vt)
c) What is the wavelength of the oscillation along the gutter (λ)
d) If the gutter has a length L. What is the wave number (L/λ)

Solutions 2

1) Equation of motion $\rightarrow x = A\cos\omega t$

$$v = \frac{dx}{dt} = -A\omega\sin\omega t$$



2/

$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1-\frac{\ell}{f_1} & \ell \\ \frac{-1}{f_1} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1-\ell/f_1 & \ell \\ \frac{\ell}{f_1 f_2}-\frac{1}{f_1}-\frac{1}{f_2} & 1-\frac{\ell}{f_2} \end{pmatrix}$$

$$\frac{1}{f^*} = \frac{\ell}{4f^2} \quad \text{in both cases}$$

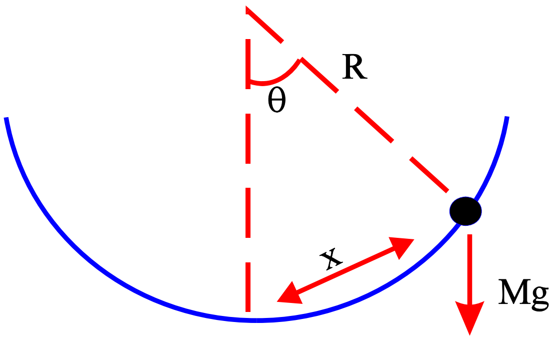
FODO Matrix $\rightarrow$

$$\begin{pmatrix} 1+\frac{\ell}{2f} & \ell \\ \frac{-\ell}{4f^2} & 1-\frac{\ell}{2f} \end{pmatrix}\begin{pmatrix} 1-\frac{\ell}{2f} & \ell \\ \frac{-\ell}{4f^2} & 1+\frac{\ell}{2f} \end{pmatrix}$$

$$\begin{pmatrix} 1-\frac{\ell^2}{4f^2}-\frac{\ell^2}{4f^2} & 2\left(\ell+\frac{\ell^2}{2f}\right) \\ \frac{-\ell}{4f^2}+\frac{\ell^2}{8f^3}-\frac{\ell}{4f^2}+\frac{\ell^2}{8f^3} & \frac{-\ell^2}{4f^2}+1-\frac{\ell^2}{4f^2} \end{pmatrix}$$

$$\begin{pmatrix} 1-\frac{\ell^2}{2f^2} & 2\ell\left(1+\frac{\ell}{2f}\right) \\ \frac{-\ell}{2f^2}\left(1-\frac{\ell}{2f}\right) & 1-\frac{\ell^2}{2f^2} \end{pmatrix} \quad \text{O.K.!!}$$

3)



Exactly like a pendulum length = R.

a) $\frac{d^2\theta}{dt^2} + \left(\frac{g}{R}\right)\theta = 0$

But $x \cong R\,\theta$ (θ is small)

$$\therefore \frac{d^2x}{dt^2} + \left(\frac{g}{R}\right)x = 0$$

$$\therefore x = A\cos(\omega t + \Phi) \qquad \omega = \sqrt{\frac{g}{R}}$$

b) S = distance along the gutter = vt.

$$x = A\cos\left(\frac{\omega}{v}s + \phi\right)$$

$$\frac{dx}{ds} = -A\frac{\omega}{v}\sin\left(\frac{\omega}{v}s + \phi\right)$$

$$\frac{d^2x}{ds^2} = -A\frac{\omega^2}{v^2}\cos\left(\frac{\omega}{v}s + \phi\right)$$

$$\frac{d^2x}{dt^2} = v^2\frac{d^2x}{ds^2}$$

$$\frac{d^2x}{ds^2} + \frac{g}{Rv^2}x = 0$$

c) Wavelength = distance traveled along the gutter whilst the ball executes 1 complete transverse oscillation: $\rightarrow\lambda$

$$\frac{d^2x}{ds^2} + \frac{g}{RV^2}x = 0$$

Wavelength, $\lambda = 2\pi/\omega$

$$\therefore \lambda = 2\pi V\sqrt{\frac{R}{g}}$$

d) Wave number = $\frac{L}{\lambda} = \frac{1}{2\pi}\frac{L}{\sqrt{v^2R/g}}$ (Q Value for a synchrotron)

Chapter 3 Lattices and Twiss parameters

First a quick revision

Our storage ring consists of dipoles to provide a closed path (nominally circular), quadrupoles to provide transverse focusing, and drift spaces. Each particle executes betatron oscillations, with it's own amplitude and phase, which are different for each particle.
The transverse motion we represent on a "Phase plot" exactly as for the pendulum example, but the focusing from the quadrupoles varies around the ring, in fact outside the quadrupoles there is no restoring force at all, and this is going to be a complication. How does this strong variation in focusing strength affect our particle motion?

The first step was to use matrices to describe each element in the machine, in order to accommodate the discrete focusing. Now we must combine these matrices with the equation for the transverse motion of our particle, for which, until now, we have assumed a constant focusing or restoring force.

We can write a general equation for transverse motion in our storage ring:-

$$\frac{d^2x}{ds^2} + K(s)x = 0 \quad \text{(use "s" not "t")}$$

K(s) is the focusing strength which is now a function of "s" and is no longer constant. This equation is called Hills equation and if we can solve it all our troubles will be over !!!..........Well maybe. Remember our gutter, now we are varying the shape of the gutter as we move around the ring (but variations are slow) See figure 1.

It is important to note that the overall oscillation amplitude depends on the initial conditions, i.e. how the motion of the ball started, but that the phase advance of it's oscillation in the gutter and the amplitude modulation in the gutter is determined solely by the shape of the gutter and not by the initial conditions. The same will be true for the particles in our accelerator or storage ring.

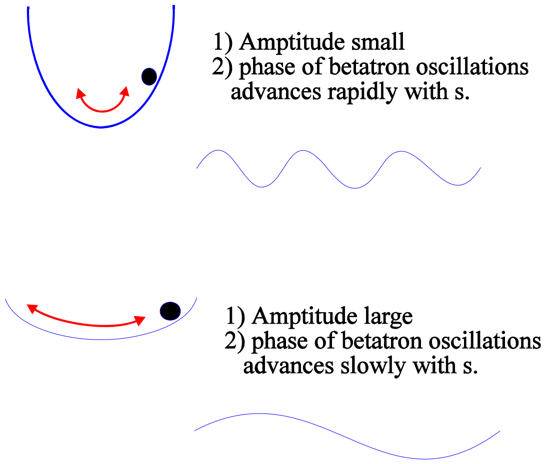


Figure 1: Relationship between oscillation amplitude and phase advance for a ball rolling along a variable radius gutter

So let's guess a solution for the equation of motion of this ball in a variable radius gutter or a pendulum of varying length. Changing the length of the pendulum is equivalent to altering the restoring or focusing force. For a long pendulum the oscillation amplitude is large and the phase advance of the oscillation is slow. For a shorter pendulum the amplitude decreases but the phase advance of the oscillation gets faster. In this way modulating the restoring or focusing force modulates the amplitude and phase advance of the oscillation. Here I have to use some help as I already know the answer, but I will try to simplify things as much as possible. Imagine a solution of the form:-

$$x = \sqrt{\varepsilon.\beta(s)}\cos\left(\Psi(s) + \phi\right)$$

where ε and ϕ are constants, which depend on the initial conditions.
$\beta(s)$ = the amplitude modulation due to the changing focusing strength.
$\psi(s)$ = the phase advance, which also depends on focusing strength.

Now if we imagine our gutter of varying radius, then we can see that if the radius is large (i.e. the focusing/restoring force is very weak) then the oscillation amplitude, $\beta(s)$, will be large and the phase advance for the oscillation, $\psi(s)$, will be slow (small). Conversely if the radius is small (i.e. the focusing force is strong) then the oscillation amplitude, $\beta(s)$, will be small and the phase advance for the oscillation, $\psi(s)$, will be fast (large). This leads to the conclusion that the rate of change of phase of the betatron oscillation around ring is proportional to the inverse of the particle's oscillation amplitude. This leads us to the following conclusion:-

$$\frac{d\psi(s)}{ds} \alpha \frac{1}{\beta(s)}$$

Now let's try and calculate the particle's position and angle (x and x'):-

$$x = \sqrt{\varepsilon\beta(s)}\cos(\psi(s)+\phi) = \text{position}$$

$$\text{and} \frac{dx}{ds} = x' = \text{angle} = -\sqrt{\varepsilon\beta(s)}\frac{d\psi(s)}{ds}\sin(\psi(s)+\phi)$$

$$x' = -\sqrt{\frac{\varepsilon}{\beta(s)}}\sin(\psi(s)+\phi)$$

Figure 2 shows a plot in phase space of these x and x' values over many turns for a single particle observed at one position, s, around the accelerator, where, of course, β is constant.

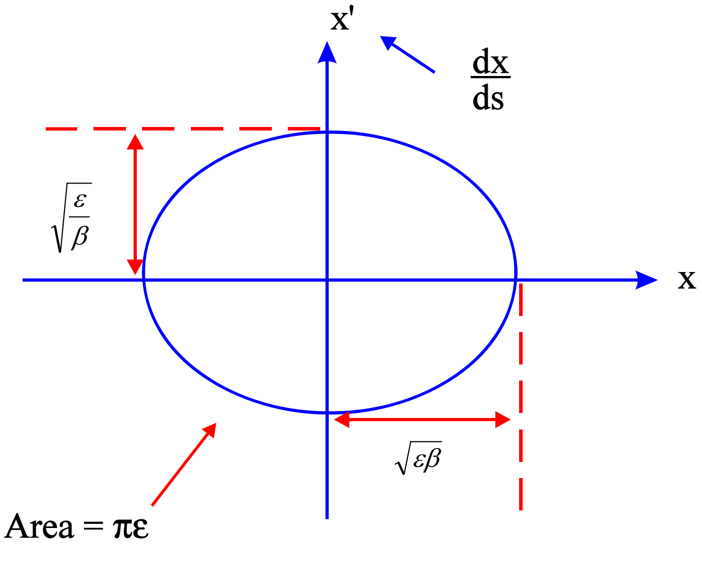


Figure 2: Transverse phase space plot for a single particle

ε is called the transverse emittance with units of M.rads, and is determined solely by initial conditions. However, as we move around the ring, $\beta(s)$ varies and alters the shape of the ellipse, but the area of the ellipse remains constant all around the accelerator. In fact to be rigorous we should define the emittance slightly differently. If we were able to observe all the particles at single position on one turn and measure both their position and angle, we would see a large number of points on our phase space plot, each corresponding to a pair of x,x’ values for each particle. See figure 3.

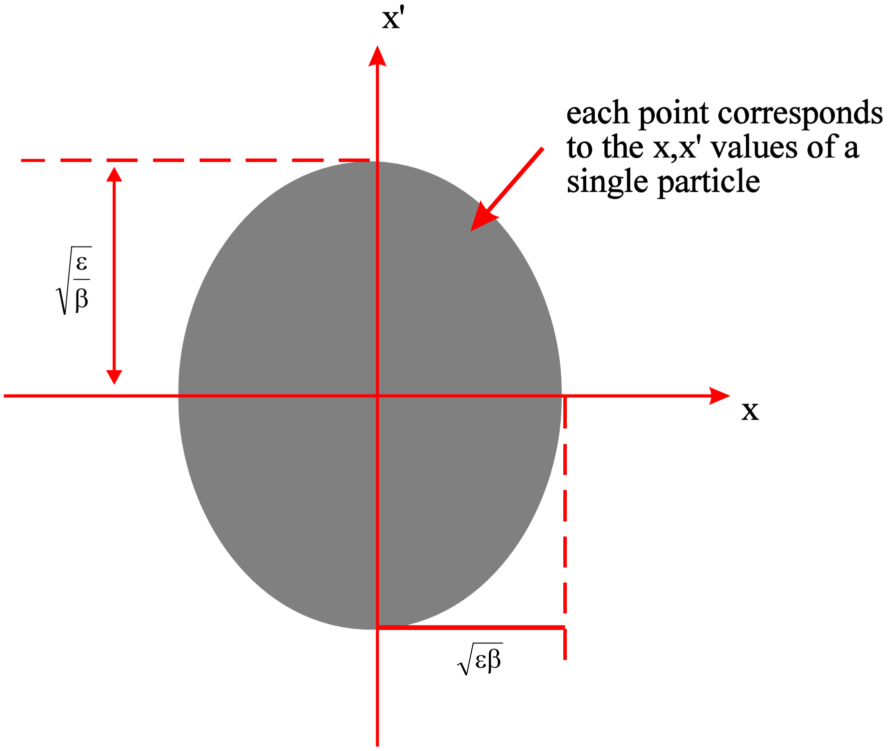


Figure 3: Transverse phase space plot for a large number of particles.

The emittance is the area of the ellipse which contains a certain percentage of the points or particles. We define a “95%” emittance as the area of ellipse which contains “95%” of the particles, or a 100% emittance as the area which contains all the particles. Therefore when talking about a transverse emittance one must be sure of the definition being used. The projection of this ellipse onto the *x* axis will give the transverse size of beam. Therefore, the beam dimensions will vary as we move around the ring as $\beta(s)$ varies. See figures 4 and 5. Remember that the initial size of the ellipse, the emittance, is determined by the initial beam conditions, and the variation of the overall beam dimensions or envelope is determined by the variation of $\beta(s)$

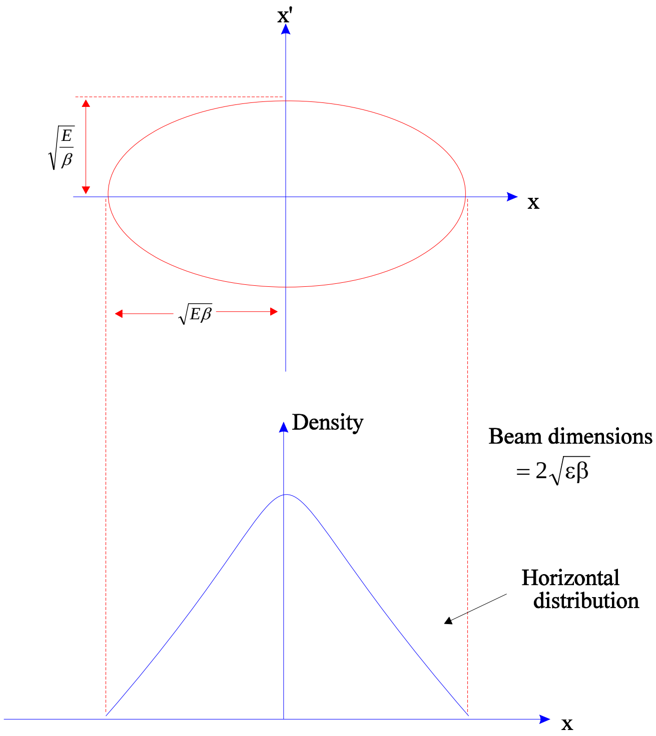


Figure 4: Phase space and transverse beam distribution for large β

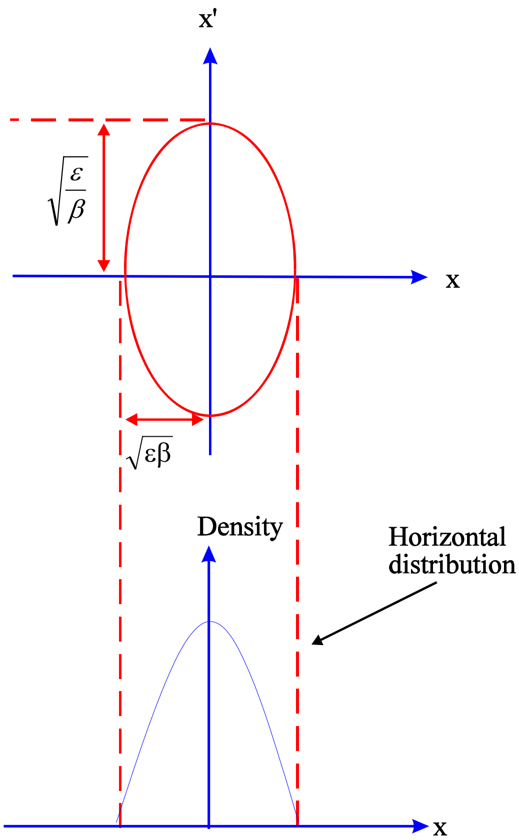


Figure 5: Phase space plot and transverse beam distribution for small β

This change in beam dimensions would seem reasonable if we remember the way we first looked at quadrupoles as optical lenses. See figure 6:

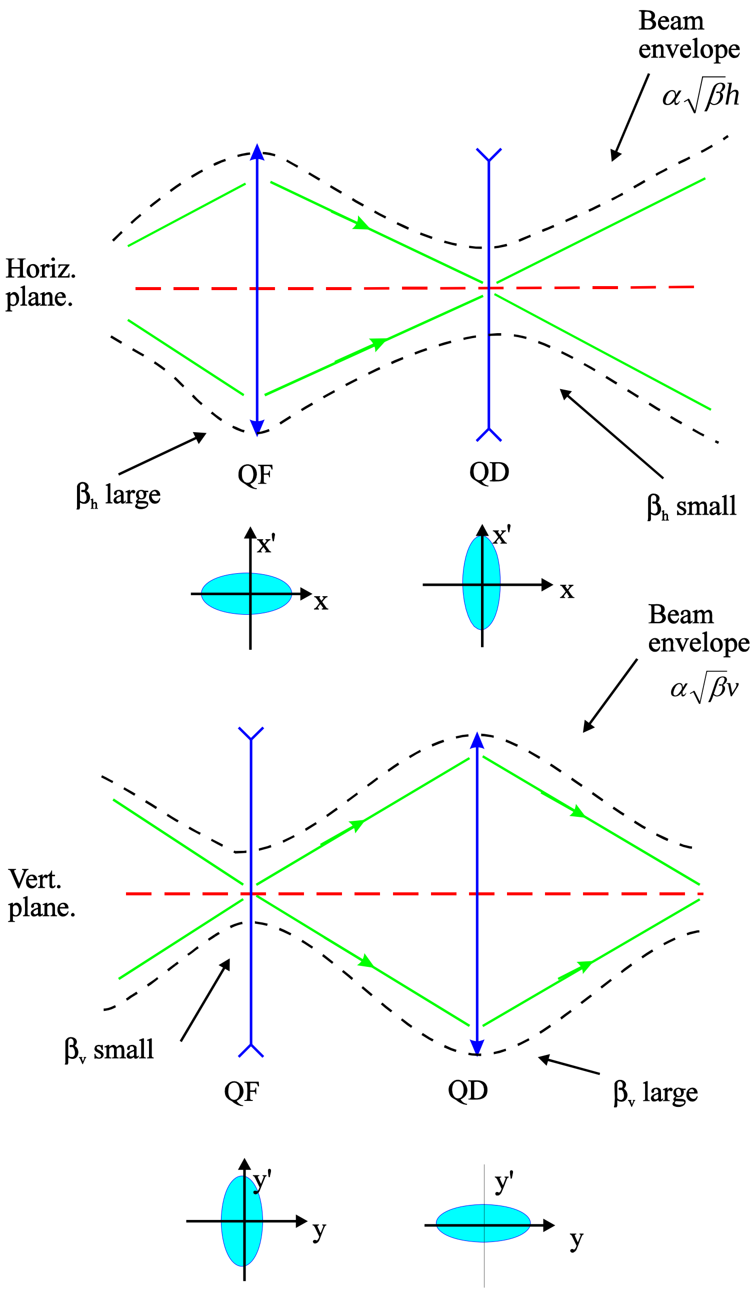


Figure 6: Beam envelope variation passing through a pair of quadrupoles

β_h and β_v will have changing values as we move around the machine, and will determine the maximum oscillation amplitude for a given emittance as the particles move around the machine. Similarly, the functions $\beta_h(s)$ and $\beta_v(s)$ determine the beam envelope in our storage ring. These functions are themselves determined by the focusing forces acting on the particles i.e. by the quadrupoles. How do we calculate β_h , β_v for a given layout of quadrupoles i.e. a given “Lattice”?

Remember also that the total number of betatron oscillations / turn = Q. But the equation of motion for our particle is :-

$$x = \sqrt{\varepsilon\ \beta(s)}\cos(\psi(s) + \phi)$$

$\psi(s)$ is the phase advance of the betatron oscillation, and therefore, since the tune, Q, of the machine is given by the change in phase of the betatron oscillation, $\Delta\psi(s)$, over a complete turn:-

$$Q = \frac{\Delta\psi(s)}{2\pi}$$

However we have already expressed the change in phase $\Delta\psi(s)$ as:-

$$\frac{d\psi(s)}{ds} = \frac{1}{\beta(s)}$$

$$\therefore \Delta\psi(s) = \int_o^s \frac{ds}{\beta(s)} \leftarrow \text{over 1 turn}$$

Remember in the last chapter we used a TRANSPORT MATRIX to move from one point on the (x,x') phase space plot to another point.

The elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of this matrix are fixed by the magnets through which the beam passes. We can also express (x,x') as solutions to Hill's equation.

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

Now for simplicity let $\beta = \omega^2$ and now we must allow β to vary rapidly with (s) i.e. $\frac{d\omega}{ds}$ or $\frac{d\beta}{ds}$ are not small and are no longer negligible. Now our "guessed" solutions to Hill's equation become:-

$$x_{(s)} = \sqrt{\varepsilon}\omega(s)\cos(\psi(s) + \phi)$$

$$x^1_{(s)} = \sqrt{\varepsilon}\frac{d\omega(s)}{ds}\cos(\psi(s) + \phi) - \frac{\sqrt{\varepsilon}}{\omega(s)}\sin(\psi(s) + \phi)$$

To move from any point s1, to another point s2 around our machine we have a transport matrix of the form:-

$$\begin{pmatrix} x_{(s2)} \\ x'_{(s2)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{(s1)} \\ x'_{(s1)} \end{pmatrix}$$

If we substitute from above for x_{s1}, x_{s1}', x_{s2} and x_{s2}' into the above matrix equation we will find four equations, which will allow us to find the individual matrix elements a, b, c, and d, in terms of β etc. This is very complicated. So we include one piece of information we have not yet used, this is a storage ring, i.e. it closes upon itself. Now we can simplify things considerably by setting (s2)=(s1) i.e. by doing our calculation for a complete turn around the ring. In this case :-

$\beta(s_2) = \beta(s_1)$ -- β is a closed function. (it is "continuous")

$\beta'(s_2) = \beta'(s_1)$

Remember that $\left(\beta' = \frac{d\beta}{ds}\right)$ and $\left(\omega' = \frac{d\omega}{ds}\right)$

Also we will invent special functions, called **TWISS PARAMETERS**

$$\alpha = {-\beta'}/{2} = -\omega\omega'$$

$\beta = \omega^2$ ←(this one we have already)

$$\gamma = \frac{1+\alpha^2}{\beta}$$

Finally we will define:-

$\mu = \Delta\psi s$ = phase change of betatron oscillation over 1 turn.

Therefore:- $Q = \frac{\mu}{2\pi}$

Now our transport matrix becomes:-

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

This Matrix describes a complete turn around our ring. It will vary depending on the starting position s. If we start at any point and multiply all of the matrices representing each element all around the ring back to our starting position, then we can calculate α, β γ and μ for this point, which will give us $\beta(s)$ and Q. If we repeat this many times for many different initial positions (s) we can calculate our lattice parameters for all points around the ring. Computer codes calculate these "lattices" of α β γ and μ.

Now we see how $\beta(s)$ varies around the ring. μ and Q do not depend on the starting point and they will be the same for each calculation, but we will be able to calculate the change in betatron phase, i.e. $d\mu$, from one element to the next. So we can vary the lengths, positions and strengths of the individual elements in the computer code to obtain the desired beam dimensions or envelope ($\sqrt{\beta(s)}$) and the desired Q in each plane. Remember $\beta(s)$ μ, and Q, are related. Often a machine is made of many identical sections (or periods). In this case we only need to calculate a single period not the whole machine, as the functions $\beta(s)$ and $d\mu$ will repeat themselves for each identical section.

Figures 7 and 8 and table 1 show the results of a lattice calculation for a real accelerator. It is a Booster synchrotron with 20 FODO cells, which accelerates electrons from 120 MeV/c to 3 GeV/c, and is in operation at the Stanford Synchrotron Radiation Laboratory. Since it is a very simple machine we will use this accelerator as an example for various calculations in this course.

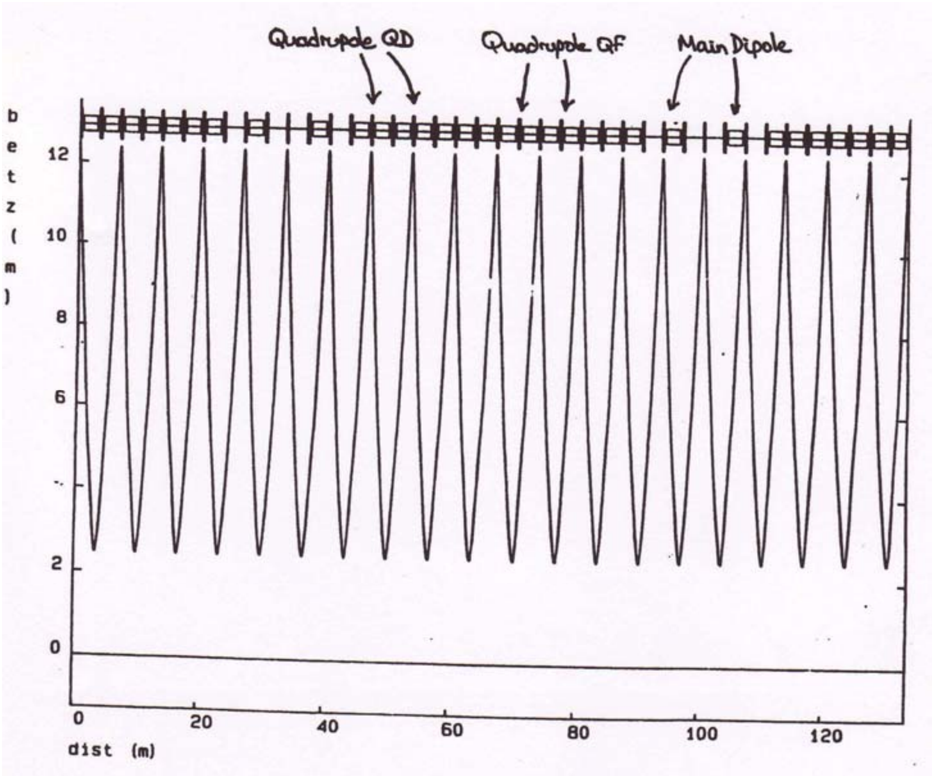


Figure 7: Vertical β function plotted as a function of (s) for our booster, notice the exact 20-fold periodicity, due to the 20 FODO cells

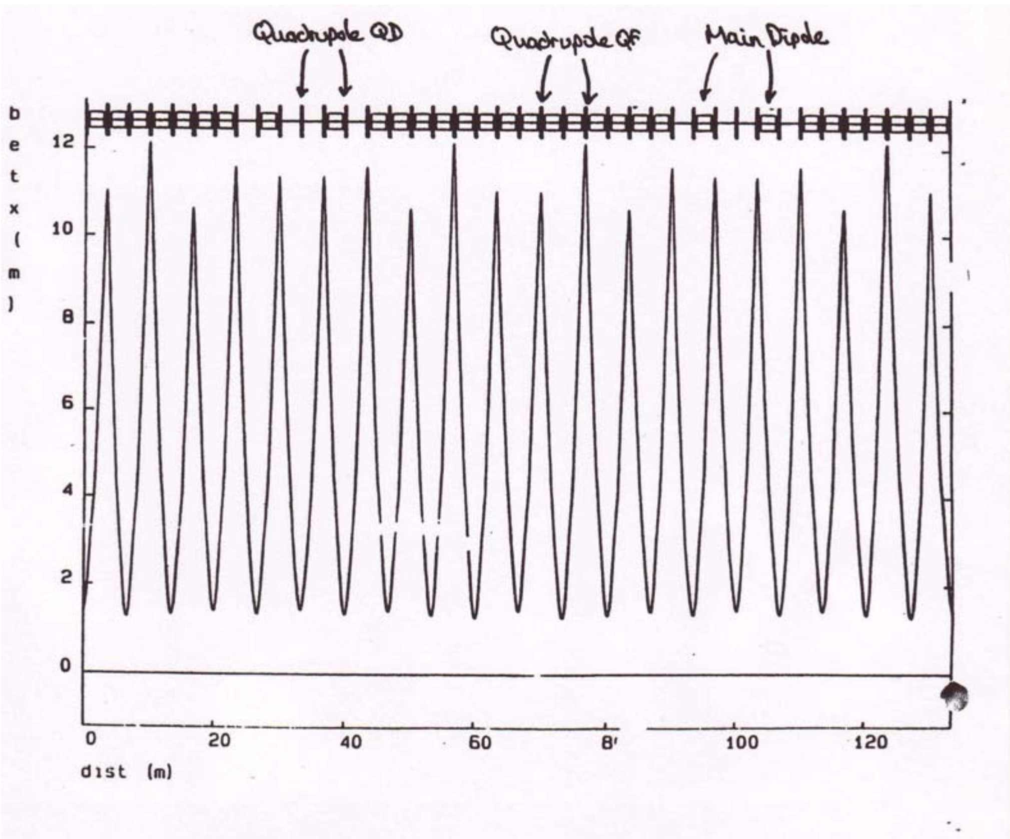


Figure 8: Horizontal β function plotted as a function of (s). Notice that the 20-fold periodicity is not quite exact.

Table 1: Lattice parameters for our “Booster”

Lattice type	FODO
Magnet structure	separated function
Cell length (m)	6.683
Total number of dipoles	32
Total number of FODO cells	20
Number of empty straight sections	8
Max. value of horiz. and vert. β functions (m)	12.2 / 12.4
Max. value of dispersion function (m)	1.69
Horizontal and vertical tunes (Q)	6.25 / 4.175
Momentum compaction factor	0.03349

Exercises 3

1) At 3 GeV in the Booster, the field in the main dipole is 8.35 KG. Each dipole is 2.35 metres long.

a) By what angle does each dipole bend the beam?
b) How many dipoles are there?
c) What will be the field strength at injection ? (120 MeV)

2) The QD quadrupoles in the Booster have a field gradient K= 130 KG m^{-1} at 3.0 GeV. They are 30 cm long.

a) What is their nomalised gradient (k)
b) What is their focal length (f)

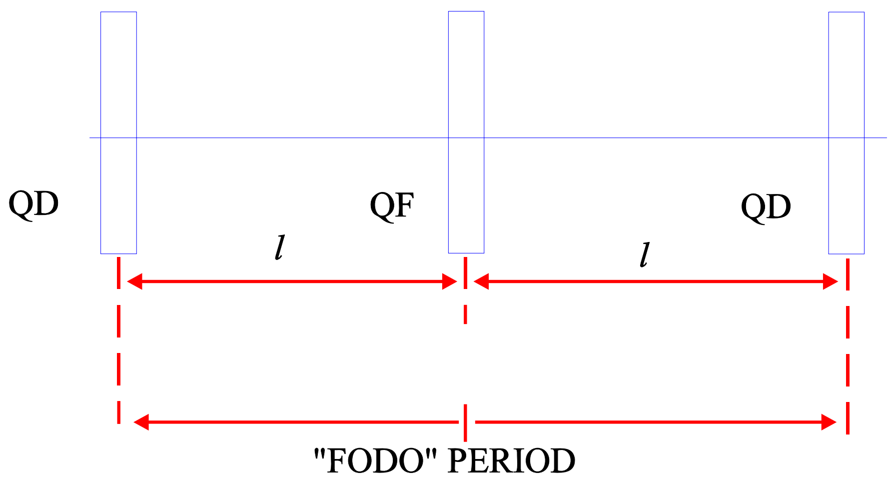
3) The QF quadrupoles in the Booster have a field gradient K = 133 KG m^{-1} at 3.0 GeV. They are 38 cm long, repeat 2(a) , 2(b) for QF quadrupoles.

4) Given that the matrix for a FODO period must have the form :-

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

a) By equating terms in this matrix with those found in question (2) of the exercises in chapter 2, find expressions for $\cos\mu$ and $\beta\sin\mu$ in terms of l and f.

In the booster there are 20 “FODO” periods, each period looks like:-



where l = 3.34 metres.

b) Using the value for f found in question (2) for a QD quadrupole calculate μv for one "FODO" period. (Assume all quadrupoles have the same focal length)

c) What is Qv for a complete turn?

d) What is βv at the centre of QD quadrupoles (beginning of the "FODO" period)

e) Repeat (b), (c) and (d) using the focal length of a QF quadrupole from question (3) and calculate μh, Qh, and βh at the centre of a QF quadrupole.

5. Why does the horizontal β function in figure 8 not follow exactly the 20-fold symmetry of the 20 FODO periods as the vertical β function does?

Solutions 3

$$\Theta = \frac{\ell B}{(B\rho)} \qquad B\rho = 33.356p \leftarrow \quad GeV/c$$

1) a.

$$\Theta = \frac{\ell B}{33.356 \times 3} = \frac{2.35 \times 8.35}{33.356 \times 3} = 0.196 \qquad \text{radians}$$

b. 1 complete turn requires Σθ = 2π radians.
therefore we need $\frac{2\pi}{0.196}$ = 32 dipoles

c.

$$\Theta = \frac{\ell B}{33.356 \times p} = 0.196 \qquad \text{radians}$$

$$\therefore \text{at } 120 \text{ MeV/c} \;\; B = \frac{0.196 \times 33.356 \times 0.120}{2.35} = 0.334\,\text{kG}$$

2) a. $k = \frac{K}{B\rho} = \frac{130}{33.356 \times 3} = 1.3\text{m}^{-2}$

QD

b. $f = \frac{1}{\ell k} = \frac{1}{1.3 \times 0.3} = 2.564\text{m}$

3) a. $k = \frac{K}{(B\rho)} = \frac{133}{33.356 \times 3} = 1.33\text{m}^{-2}$

QF

b. $f = \frac{1}{\ell k} = \frac{1}{1.33 \times 0.38} = 1.979\text{m}$

4) a. Adding terms "a and d" $\rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

gives

$$\cos\mu + \alpha\sin\mu + \cos\mu - \alpha\sin\mu = 2\left(1 - \frac{\ell^2}{2f^2}\right)$$

$$\cos\mu = \left(1 - \frac{\ell^2}{2f^2}\right)$$

from term "b"

$$\beta\sin\mu = 2\ell\left(1 + \frac{\ell}{2f}\right)$$

b. f for a QD = 2.564 m. and l = 3.34 m.

$$\therefore \cos\mu = 1 - \frac{3.34^2}{2\times 2.564^2} = +0.1354$$

μ = 1.435 radians

c. There are 20 periods, therefore over 1 turn the complete phase advance = 1.435×20 radians

But Q = total phase advance/2π

$$\therefore \;\; Qy = \frac{1.435\times 20}{2\pi} = 4.57$$

d.

$$\beta \sin\mu = 2\ell\left(1+\frac{\ell}{2f}\right)$$

$$\beta y = 6.68\times\left(1+\frac{3.34}{2\times 2.564}\right)\times\frac{1}{\sin(1.435)}$$

βy = 11.1 metres at QD quadrupoles.

For QF quadrupoles.

b. f = 1.979m. l = 3.34m

$$\cos\mu = 1-\frac{3.34^2}{2\times 1.979^2} = -0.424$$

$\therefore \mu$ = 2.01 radians.

c) Qx = 6.40 $\left(\frac{2.01\times 20}{2\pi}\right)$

d) $$\beta x = 6.68\times\left(1+\frac{3.34}{2\times 1.979}\right)\times\frac{1}{\sin(2.01)}$$

βx = 13.6 metres at QF quadrupoles.

5. There are several missing dipoles in the ring, which have no focusing effect in the vertical plane, but have a small effect in the horizontal plane. This shows up as a slight modulation of the horizontal β function.

Chapter 4: Tune shifts, closed orbits, dispersion and chromaticity

In this chapter we will introduce perturbations to the accelerator we calculated in the last chapter. Firstly we will introduce quadrupole errors, which lead to tune shifts, and tune correction, then dipole errors for closed orbits, orbit correction and orbit bumps, and finally we will examine the effect of adding a finite momentum spread to the particles, which will lead to the ideas of dispersion and chromaticity.

Quadrupole errors

The Twiss matrix for our "FODO" period is given by :-

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Imagine adding a small quadrupole error, which we will describe as having strength dK and length ds. This extra quadrupole, assuming that we have added a small QF quadrupole, will modify the FODO lattice, and add a horizontal focusing term, which is given by :-

$$\begin{pmatrix} 1 & 0 \\ -\mathrm{dk}\,\mathrm{ds} & 1 \end{pmatrix} \quad \text{where} \quad \mathrm{dk} = \frac{\mathrm{dK}}{(\mathrm{B}\rho)} \quad \text{and} \quad f = \frac{(\mathrm{B}\rho)}{\mathrm{dKds}}$$

Now the new Twiss matrix representing the modified lattice is:-

$$\begin{pmatrix} 1 & 0 \\ -\mathrm{dk}\,\mathrm{ds} & 1 \end{pmatrix}\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \qquad (1)$$

This extra quadrupole will modify the phase advance μ for the FODO cell. However the new Twiss matrix must be the same as the Twiss matrix for our modified FODO cell, in which we replace the original phase advance μ with the modified phase advance μ_1, where:

$\mu_1 = \mu + d\mu$,

$d\mu$ = change in phase advance due to the extra focusing term

If $d\mu$ is small then we can ignore changes in β etc. So the new Twiss matrix can be evaluated by multiplying out expression (1) and equating the resulting elements with the Twiss matrix with μ replaced by μ_1.

$$\therefore \begin{pmatrix} \cos\mu_1 + \alpha\sin\mu_1 & \beta\sin\mu_1 \\ -\gamma\sin\mu_1 & \cos\mu_1 - \alpha\sin\mu_1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -dk\,ds(\cos\mu + \sin\mu) - \gamma\sin\mu & -dk\,ds\beta\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Combing the first and third terms of the two above matrices we get:-

$$2\cos\mu_1 = 2\cos\mu - dk\ ds\,\beta\sin\mu$$

$$\therefore\ 2\cos(\mu + d\mu) = 2\cos\mu - 2\sin\mu\, d\mu = 2\cos\mu - dk\ ds\,\beta\sin\mu$$

$$\therefore\ 2\sin\mu\ d\mu = dk\ ds\,\beta\sin\mu$$

$$\therefore\ d\mu = \frac{1}{2}\,dk\,ds\,\beta \qquad \left(dQ = \frac{d\mu}{2\pi}\right)$$

So the change in Qh, dQh, for one error, dk, is given by:-

$$dQ_h = +\frac{1}{4\pi}\,dk\ ds\,\beta$$

For many errors, we must sum around the machine.

$$dQh = \frac{1}{4\pi}\Sigma \quad dk(s)\,ds\,\beta(s) \rightarrow \frac{1}{4\pi}\int dk(s)\,\beta(s)\,ds$$

Remember there will also be a vertical effect. If we rewrite the initial matrix, *dkds* now represents a defocusing effect and therefore f is negative.

$$\therefore \quad \begin{aligned} dQv &= -\frac{1}{4\pi}\beta_v\, dk\ ds \\ dQh &= +\frac{1}{4\pi}\beta_h\, dk\ ds \end{aligned}$$

Suppose at the same time we allow extra QD quadrupole, and follow exactly the same reasoning, we get the overall change in Qh and Qv for a QF, QD pair :-

$$dQv = +\frac{1}{4\pi}\beta_v\ dk\,ds - \frac{1}{4\pi}\beta_v\ dk\,ds$$

$$dQ_H = -\frac{1}{4\pi}\beta_h\ dk\,ds + \frac{1}{4\pi}\beta_h\,dk\,ds$$

Let dk_F = dk for QF and dk_D = dk for QD, and let:-

β_{VF}} represent β at the QF, and β_{VD}} represent β at the QD
β_{HF}} β_{HD}}

$$\begin{pmatrix} dQv \\ dQh \end{pmatrix} = \begin{pmatrix} \frac{+1}{4\pi}\beta_{VD} & \frac{-1}{4\pi}\beta_{VF} \\ \frac{-1}{4\pi}\beta_{HD} & \frac{+1}{4\pi}\beta_{HF} \end{pmatrix} \begin{pmatrix} dk_D ds \\ dk_F ds \end{pmatrix}$$

This matrix "M" relates the change in the tune of the machine to the change in focal length (or magnetic field) of the quadrupoles.

We could also invert "M" to calculate change in quadrupole field needed for a given change in tune, but take care with the sign of dk_d and dk_f. As a rule of thumb:-

Increasing k_D → decreases H tune
increases V tune

and vice versa.

Increasing k_F → increases H tune
decreases V tune

<u>Dipole error.</u>

Here we will introduce two small changes in an attempt to simplify the presentation of the ideas.
Firstly for betatron motion we have plotted phase space as X. v .X' and Y. x .Y'. For orbits we will use "nomalised" phase space, this means we will plot Y. v .(βY').
In Figure 1 the motion of the particles on the normalised phase space plot is always a circle, but the area of the circle is not constant as we move around the machine. Secondly, instead of using "s" to describe the longitudinal position around the machine we will use θ. See figure 2.

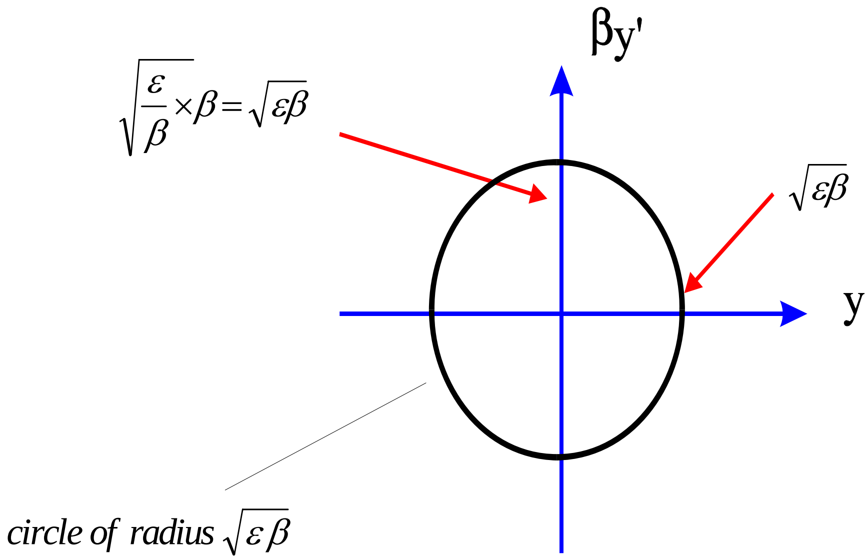


Figure 1: Normalised vertical phase space

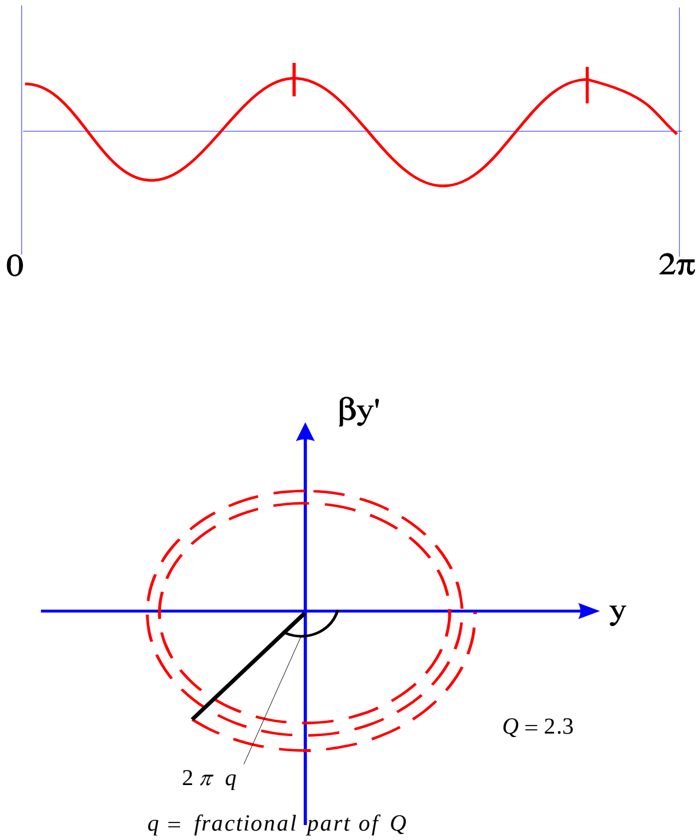


Figure 2: Trajectory and normalised phase space plot for Q = 2.3

In figure 2, for a single turn around the machine, we will move 2.3×2π radians around the normalised phase space plot. Therefore the phase advance over a complete turn is 2πq, where q is the fractional part of Q.

Now let's get back to our discussion of dipole errors with a definition. The "Closed orbit" is defined as the average trajectory of all of the particles in our machine, or the path of a single particle averaged over many turns. It is the path about which all particles perform their individual betatron oscillations. A particle with very small betatron oscillations amplitude will almost follow this closed orbit. If we plot the trajectory of this particle in "normalised" phase space, then we get a single point at the centre of the diagram, which is not very exciting. See figure 3.

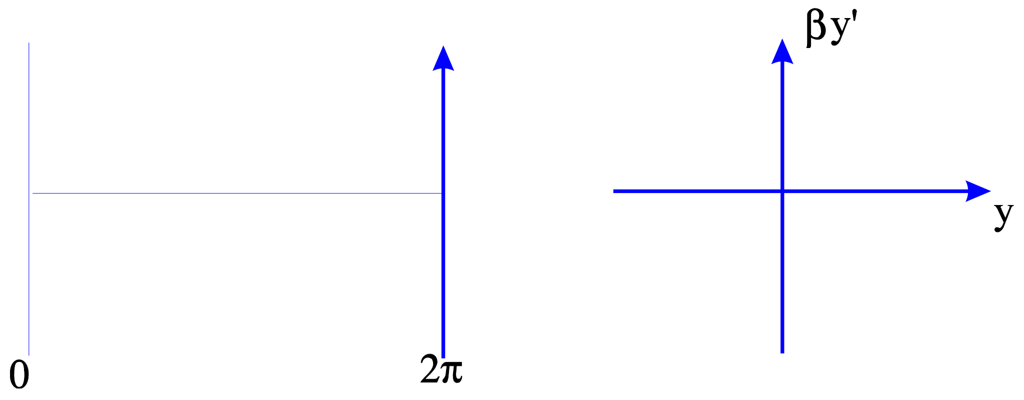


Figure 3: Trajectory and phase space plot for a particle circulating exactly on the closed orbit

Now introduce a vertical dipole error at the middle of the trajectory ($\theta = \pi$) in figure 3. This dipole gives a vertical deflection 'd', which the particle undergoes on each turn. See Figure 4.

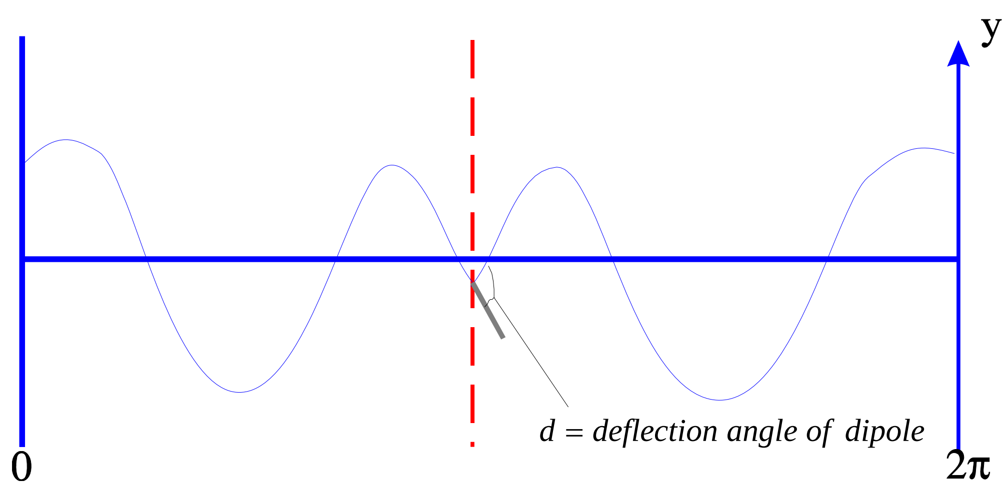


Figure 4: Modified trajectory for a single dipole error

This trajectory must close upon itself, since, by definition, it is a "closed orbit". The resulting curve in figure 4 looks like a betatron oscillation, but it is not, as there is a sharp phase change at the position of the kick ($\theta = \pi$). However, the focusing forces, which create this trajectory, are exactly those, which give rise to the betatron oscillations and, therefore the total number of transverse oscillations per turn for our new closed orbit will be Q. Under these conditions it is very tempting to write the displacement "y(s)" of our closed orbit as a function of "s" around the ring as :-

$$y(s) = \sqrt{\varepsilon \beta(s)} \cos(Q\Theta(s))$$

Here the "ε" is definitely not an emittance, but it is a term, which will give us the overall amplitude of the closed orbit deformation due to the dipole error, and as such must be related to the strength of the dipole kick. The β(s) function again describes the form of the amplitude modulation due to the quadrupoles.

How does this orbit look on our "nomalised" phase space. See figure 5.

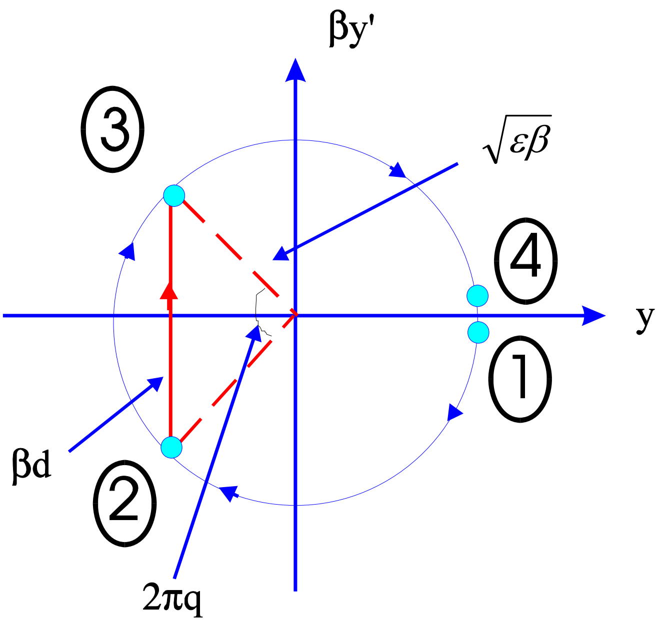


Figure 5: Deformed orbit on a normalised phase space plot

In figure 5 the particle starts at $\theta = 0$, at position ①, and it's phase advances by 1.33×2π, when we reach ②. Now the angle changes by d (βd in figure 5) and we reach ③. Finally the phase advances by 1.33×2π to bring us to ④ $\theta = 2\pi$, where the whole pattern starts again.
From Figure 5 by simple trigonometry:-

$$\sin(\pi Qv) \quad = \frac{\beta d}{2\sqrt{\varepsilon\beta}}$$

$$\sqrt{\varepsilon} = \frac{\sqrt{\beta} d}{2\sin(\pi Qv)}$$

Combine this with our previous guess as to a solution for our closed orbit, and we obtain the displacement at some position “s” as y(s):-

$$y(s) = \sqrt{\beta\beta(s)}\frac{d}{2\sin(\pi\, Qv)}\cos(Qv\Theta(s))$$

where β = β function at the dipole “error”
$\beta(s)$ represents the β function as a function of s
Qv = Vertical tune

Now d = angle error = $\frac{\ell\Delta B}{(B\rho)}$ where $\begin{bmatrix} \text{dipole} \quad \text{length} = \ell \\ \text{Field} = \Delta B \end{bmatrix}$

$$\therefore y(s) = \frac{\sqrt{\beta\beta(s)}}{2\sin(\pi Qv)}\cos(Qv\Theta(s)) \times \frac{\ell\,\Delta B}{B\rho}$$

As θ varies from 0 to 2π this describes the amplitude of the closed orbit distortion due to a single dipole error of length *l* and field ΔB. If β at the error is large, then the error is important, and if $\beta(s)$ is large then distortion is important. This large distortion will reduce the aperture available for the beam and, therefore it is important to minimise these orbit distortions. This means we will need some method of orbit correction.

Normally the overall closed orbit will be the sum of many small errors, but where do these errors come from?

Horizontal plane $\rightarrow$ Dipoles - wrong length - wrong field.
Vertical plane $\rightarrow$ Dipoles - tilts.
Both planes $\rightarrow$ Stray fields in drift spaces.
Both planes $\rightarrow$ Displaced Quadrupoles $\leftarrow$ (important)

The last category above is the most important. Imagine a QF displaced horizontally. There is no field in the geometrical centre of a quadrupole, but if the centre of the quadrupole is displaced with respect to the beam axis, then the beam sees a horizontal deflection due to this displacement. β_H is large here (in QF) and therefore this error is important! So we must correct these errors. This can be done by:-

1/ Moving the quadrupoles to correct misalignments, tilts etc. This works for alignment errors, but is time consuming and will never correct field errors or stray fields.

2) Using small dipoles around the ring to cancel effects of unwanted errors.
Good places to position these correctors are close to QF quadrupoles for horizontal correctors, and close to QD quadrupoles for vertical correctors, where the relevant β functions are largest.

<u>But how do we correct closed orbit ?</u>

First we must measure the orbit, by reading the average position from each Beam Position Monitor (BPM) over many turns. Ideally we use 4 BPM's per betatron wavelength, to define the orbit. We express the measured "error" as a "peak to peak" distortion or a root mean square "rms" distortion. The rms distortion or rms error is given by:-

$$\frac{\sqrt{\sum_{n=1}^{n=N} \Delta y_n{}^2}}{N}$$

Where N = number of BPM's and Δy_n = displacement at n_{th} BPM.

Usually there will be fewer correctors than BPM's. But each corrector will move the beam at each BPM by a certain amount $\Delta Y(s)$:-

$$\Delta Y(s) = \frac{\sqrt{\beta \beta(s)}}{2\sin(\pi Qv)} \cos(Q_v \Theta(s)) \frac{\ell \Delta B}{(B\rho)}$$

Now we can use a computer to calculate set of corrector fields which will minimize the rms orbit error.

<u>Orbit Bumps</u>

Sometimes it is useful to move the closed orbit in just one spot and not move it elsewhere. In this case we use a beam bump. Here we must be careful as changing just one dipole affects the orbit all around the machine. See figure 4. We will need a minimum of two dipole correctors to create such a localised orbit bump. Figure 6 shows an ideal two magnet orbit bump. Ideally we would like to use correctors exactly at positions ① and ② to displace the beam at position ③. What does this look like on our "normalised phase space" plot? See Figure 7.

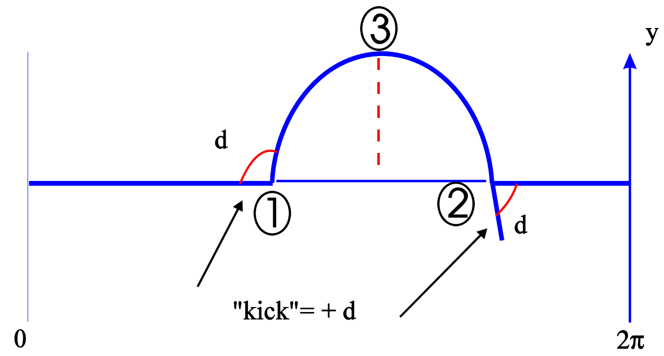


Figure 6: An ideal two magnet bump

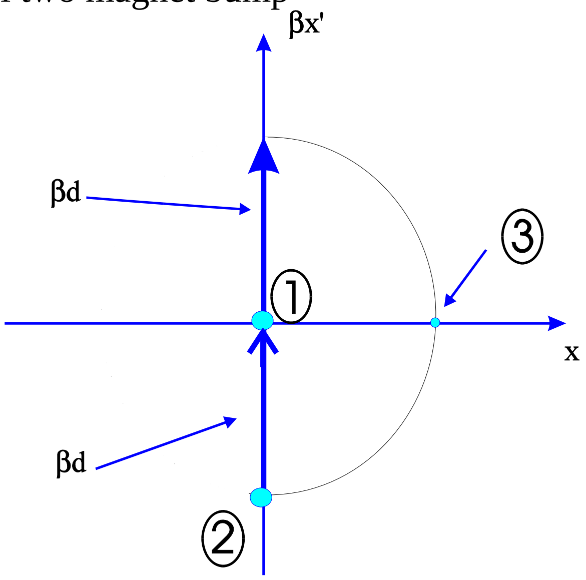


Figure 7: An ideal 2 magnet bump on a normalised phase space plot

The beam circulates at the centre of the plot until it reaches position (1), where it sees a deflection βd. Now it moves around the phase space plot to position (3), where the bump amplitude is a maximum. From here it moves to position (2), where there is another deflection βd, which replaces the beam at the centre of the plot. This only works if the phase advance from (1) → (2) is exactly π, or half a betatron oscillation period.

The amplitude of such a bump is given by Δy, where:-

$$\Delta y \;\alpha\; \sqrt{\beta_1 \beta_3}\, d$$

If β_1 and β_3 are large then we will get a large bump for a small d. Therefore again we should look at QD quads for vertical dipoles and QF quads for Horizontal dipoles, exactly as for the closed orbit correction.

For Vertical dipoles close to the QD quadrupoles in our example the vertical phase advance, μ, = 85° between QD quadrupoles. Therefore by using dipoles beside alternate quadrupoles we can obtain reasonable "two magnet bumps. However the horizontal phase advance, μ, = 115° between QF quadrupoles. Therefore if we use only 2 dipoles beside QF quadrupoles we will get a residual distortion all around the machine. See figure 8.

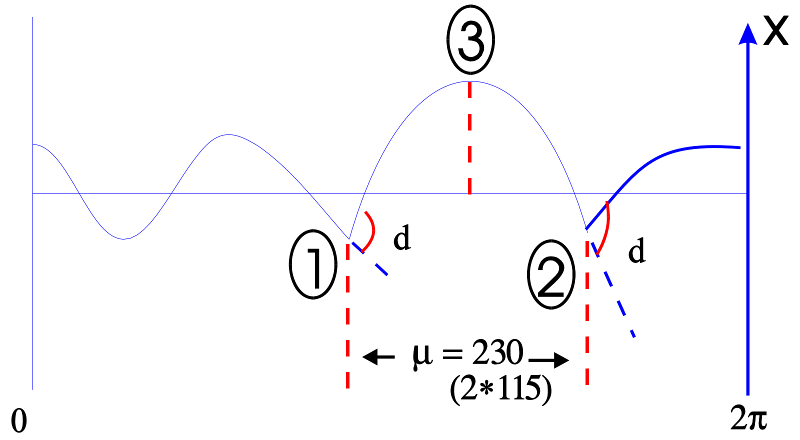


Figure 8: A two magnet bump, where the phase advance is not 180 degrees

To cancel this residual orbit distortion we add a third dipole at position ④ See figure 9.

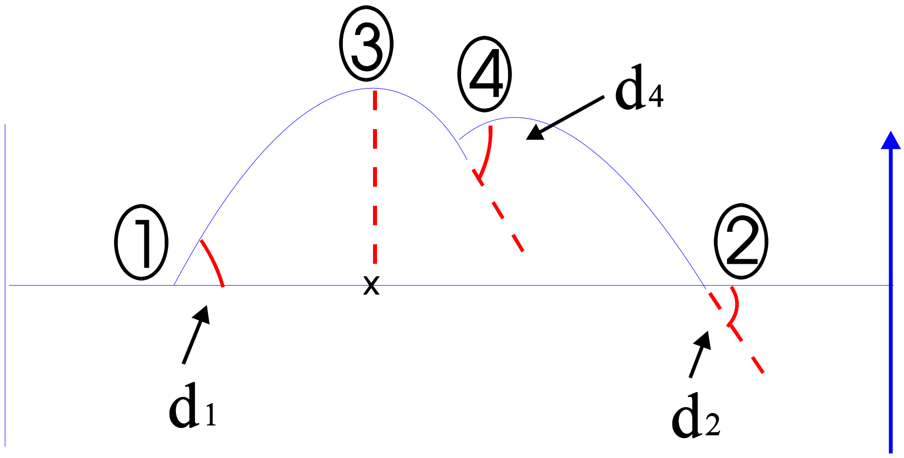


Figure 9: A "three" magnet orbit bump

Now d_1 gives the correct $\Delta\, x$ at position 3), d_4 puts beam back on axis at position 2), and d_2 corrects angle error at position 2), to put the beam back onto the ideal orbit. Figure 11 shows a phase space plot for a general "three" magnet bump.

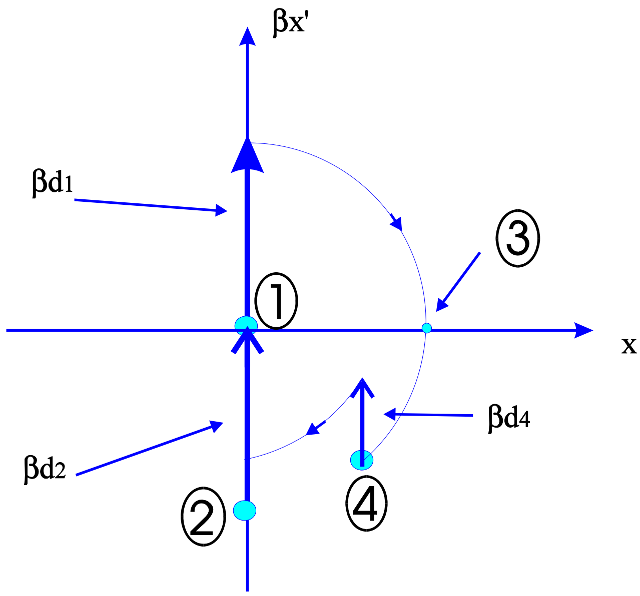


Figure 10: A general "three" magnet bump

Sometimes if we want to control position (Δx) and the angle ($\Delta x'$) at a certain point, e.g. at an ejection septum, we use 4 or even 5 dipole bumps for greater flexibility. In general a bump is just a local orbit distortion using a small (2 or more) number of dipoles.

Until now we have assumed that our beam has no energy or momentum spread ΔE or Δp. we usually write this as $\frac{\Delta E}{E}\, or \,\frac{\Delta P}{P}$. However different energy or momentum particles have different radii of curvature (ρ) in the main dipoles. Now these particles no longer pass through the quadrupoles at the same radial positions. In this case the quadrupoles themselves now act as dipoles for different momentum particles. This mean that the closed orbits for different momentum particles are different. This displacement is expressed as the dispersion function D(s) (metres). D(s) is a function of "s" exactly as β(s) is a function of "s". The displacement due to the change in momentum at any position (s) is given by:-

$$\Delta\, x\, (s) = D(s)\, \frac{\Delta P}{P}$$

$\Delta x\, (s) \rightarrow$ displacement due to the momentum spread.

D(s), the dispersion function, is calculated from the lattice, and has units of metres. Because of this dispersion function the beam will have a finite size due to it's momentum spread. Normally we have no vertical dipoles, and so D(s) = 0.0 in the vertical plane. i.e. vertical position does not depend on momentum. The change in orbit with changing momentum means that the average length of the orbit will also depend on the beam momentum. This is expressed as the momentum compaction factor, αp, where:-

$$\frac{\Delta r}{r} = \alpha p \frac{\Delta p}{p}$$

αp tells us about the change in the length or radius of the closed orbit for a change in momentum.

But the focusing strength of our quadrupoles also depends on the beam momentum, "p", remember:

$$k = \frac{dBy}{dx} \times \frac{1}{B\rho}$$

Therefore a spread in momentum, $\frac{\Delta P}{p}$, causes a spread in focusing strength, $\frac{\Delta k}{k}$

$$\therefore \frac{\Delta k}{k} = -\frac{\Delta P}{P}$$

But the betatron tune, Q, depends on k for the quadrupoles.

$$\therefore \quad \frac{\Delta Q}{Q} = \xi \frac{\Delta P}{P}$$

ξ is yet another Greek letter and is called the "chromaticity". It represents the change in betatron tune with particle momentum. Obviously, as the k for a quadrupole decreases with increasing momentum, the focal length of the quadrupoles increases with increasing momentum. Therefore, the chromaticity of an accelerator containing only dipoles and quadrupoles is always negative, i.e. the tune decreases with increasing momentum. This is called the natural chromaticity of the accelerator. We will try to calculate the relationship between tune and momentum in the next chapter.

Exercises 4

In "our" booster : $K_F = 0.222\times I$ KGM^{-1} and $L_{QF} = 0.38$ m
$K_D = 0.218\times I$ KGM^{-1} and $L_{QD} = 0.3$ m

(where I = Quadrupole current in amps.)

β_H at QF = 12.0 m. β_V at QF = 2.5 m.

β_H at QD = 1.3 m β_V at QD = 12.4 m.

1) Write out the "2 x 2" matrix that gives

$$\begin{pmatrix} dQ_V \\ dQ_H \end{pmatrix} \text{ in terms of } \begin{pmatrix} dk_D\, ds \\ dk_F\, ds \end{pmatrix}$$

2) Invert the above matrix.

3) What values of dk_D , dk_F are needed to change Qv and Q_H from the values of 4.57 and 6.40 as calculated in chapter 3) to the design values of 4.175 and 6.250.

4)What current would be needed for the above correction at 120 MeV/c and 3.0 GeV/c.

5) A QF quadrupole in the Booster is displaced horizontally by 0.5 cms, how big will the maximum orbit distortion be at QF and QD quadrupoles. (Use $k_{QF} = 1.33$ m^{-2})

6) Will this distortion change as a function of beam energy during acceleration.

Solutions 4

1) $$\begin{pmatrix} dQv \\ dQh \end{pmatrix} = \frac{1}{4\pi}\begin{pmatrix} \beta_{VD} & -\beta_{VF} \\ -\beta_{HD} & \beta_{HF} \end{pmatrix}\begin{pmatrix} dk_D ds \\ dk_F ds \end{pmatrix}$$

$$= \frac{1}{4\pi}\begin{pmatrix} 12.4 & -2.5 \\ -1.3 & 12.0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.99 & -0.2 \\ -0.1 & 0.96 \end{pmatrix}$$

2) Inverse of this matrix = $\frac{1}{0.97}\begin{pmatrix} 0.96 & 0.2 \\ 0.1 & 0.99 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0.2 \\ 0.1 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} dk_D ds \\ dk_F ds \end{pmatrix} = \begin{pmatrix} 1 & 0.2 \\ 0.1 & 1 \end{pmatrix}\begin{pmatrix} dQv \\ dQh \end{pmatrix}$$

3) Required ΔQv over 1 FODO period = - 0.395 / 20 = - 0.0199
Required ΔQh over 1 FODO period = - 0.15 / 20 = - 0.0075

$$\begin{pmatrix} dk_D ds \\ dk_F ds \end{pmatrix} = \begin{pmatrix} 1 & 0.2 \\ 0.1 & 1 \end{pmatrix}\begin{pmatrix} -0.0199 \\ -0.0075 \end{pmatrix} = \begin{pmatrix} -0.021 \\ -0.009 \end{pmatrix}$$

$$dk_D = \frac{-0.021}{L_{QD}} = -0.07 m^2$$

$$dk_F = \frac{-0.009}{L_{QF}} = -0.024 m^2$$

Remember:

$$k = \frac{K}{(B\rho)} = \frac{K}{33.356p} \qquad \left(K \text{ in } \quad KGm^{-1}\right)$$

$$\therefore dk_D = \frac{0.218 dI_D}{33.356p} \quad \text{and} \quad dk_F = \frac{0.222 dI_F}{33.356p}$$

At 0.12 GeV/c:-

$$dI_F = \frac{33.356}{0.222} \times 0.12 \times (-0.024) = -0.43\text{Amps.}$$

$$dI_D = \frac{33.356}{0.218} \times 0.12 \times (-0.07) = -1.29\text{Amps.}$$

At 3.0 GeV/c:-

$$dI_F = -10.75\text{Amps}$$

$$dI_D = -32.3\text{Amps}$$

5) Maximum orbit displacement due to a dipole error

$$= \frac{\sqrt{\beta_1\beta_2}}{2\sin\pi Q_H} \times 1 \times \frac{\ell B}{(B\rho)} \qquad \left[\text{maximum value of } \cos(Q_H\Theta) = 1\right]$$

β_1 = 12.0 metres. Q_H = 6.25
at a QD β_2 =1.3 metres.
at a QF β_2 = 12.0 metres.

"Kick" or deflection = $\frac{\ell B_y}{(B\rho)} = \frac{\ell Kx}{(B\rho)} = \ell kx.$ where $k = 1.33\text{m}^{-2}$

x = the displacement of the quadrupole, and *lkx* is independent of the beam momentum. Therefore the orbit distortion will not change during acceleration.

"Kick" = 1.33 x 0.38 x 0.005 = 2.53 x 10^{-3} radians.

Maximum displacement at QF = $\frac{\sqrt{12.0 \times 12.0} \times 2.53 \times 10^{-3}}{1.414}$ = 21.5 mm

Maximum displacement at QD = $\frac{\sqrt{1.3 \times 12.0} \times 2.53 \times 10^{-3}}{1.414}$ = 7.1 mm

Chapter 5: Chromaticity, resonances and coupling

At the end of chapter 4 we introduced the idea that the particle beam in our accelerator will have a non-zero momentum spread. The normalised focusing strength of a lattice quadrupole, k, is given by:-

$$k = \frac{1}{(B\rho)} \frac{dBy}{dx}$$

Therefore there will be a spread in focusing strength, Δk, due to the finite momentum spread, Δp. But this will mean that there will a non-zero betatron tune spread, ΔQ, across the beam, as a result of this momentum spread.

$$\frac{\Delta Q}{Q} \alpha \frac{\Delta p}{p}$$

Usually we write $\frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$. The quantity ξ is called the chromaticity and it tells us how betatron tune changes with particle momentum. But we saw last time that:-

$$\Delta Q = \frac{1}{4\pi}(\beta dkds) \quad \text{and} \quad \frac{dk}{k} = -\frac{dp}{p}$$

$$\therefore \quad \frac{\Delta Q}{Q} = \underbrace{-\frac{1}{4\pi}\left(\beta \frac{k}{Q} ds\right)}_{\text{chromaticity} = \xi} \frac{\Delta p}{p}$$

We will almost always need to correct this tune spread especially in machines with a large momentum spread. We will see why in the section on resonances. This means that we need to increase the quadrupole focusing strength for higher momentum particles, and decrease it for the lower momentum particles.

Remember in a quadrupole (By ∝ x) i.e. k is constant. Now we will add a term By ∝ x^2 . This we will obtain using a sextupole magnet. Figure 1 shows a sextupole (6-pole) magnet

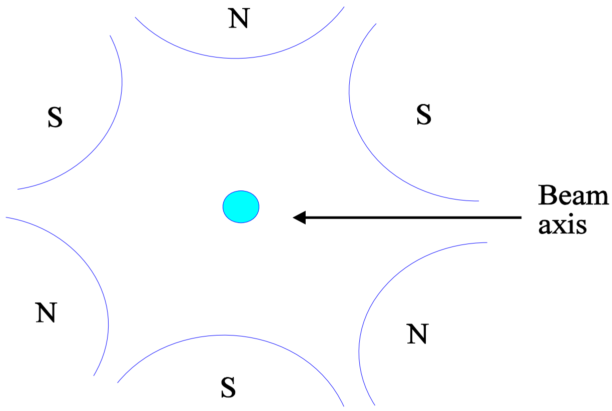


Figure 1: Schematic representation of a sextupole magnet

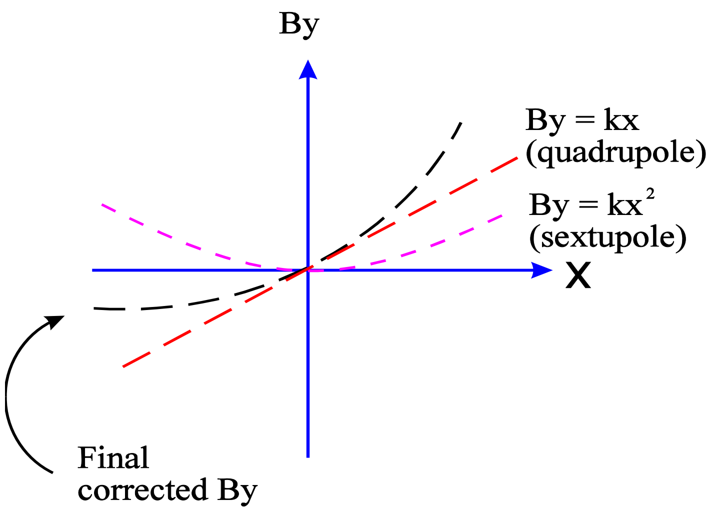


Figure 2: Magnetic fields due to a quadrupole and a sextupole.

In Figure 2 the magnetic fields due to a quadrupole and a sextupole are shown. The effect of the sextupole field is to increase the magnetic field of the quadrupoles for "positive x" particles and decrease the field for "negative x" particles. However, we have already seen that the dispersion function, D(s), describes how the radial position of the particles changes with momentum. Therefore the sextupoles will alter the focusing field seen by the particles as a function of their momentum. This we can use to compensate the natural chromaticity of the machine.

Remember in a quadrupole that $\frac{dBy}{dx}$ is constant. Now a sextupole field is defined as $\rightarrow$ $\quad By = Cx^2$

The deflection in a sextupole is = $\dfrac{\ell By}{(B\rho)} = \dfrac{\ell Cx^2}{(B\rho)}$

Now in exactly the same way as we did for a quadrupole we can define "k"

$$k = \frac{1}{(B\rho)} \frac{dB_y}{dx}$$

$$k = \frac{1}{(B\rho)} 2Cx$$

NB. k is no longer constant, it depends on "x" the transverse position of the particle. In this way the focal length of the quadrupoles is modified by the sextupoles. The change in "k", Δk, as a function of "x", is given by:-

$$\frac{\Delta k}{\Delta x} = \frac{d^2 B_y}{dx^2} \times \frac{1}{(B\rho)}$$

$$\text{But } \Delta x = D(s)\frac{\Delta p}{p} \quad \left(\text{variation of x with } \frac{\Delta p}{p}\right)$$

$$\therefore \Delta k = \frac{d^2 B_y}{dx^2} \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$$

$$\text{But} \qquad \Delta Q = \frac{1}{4\pi}\beta(s)dkds = \frac{1}{4\pi}\beta(s)\Delta k\ell$$

$$\therefore \Delta Q = \frac{1}{4\pi}\underbrace{\ell\beta(s)\frac{d^2By}{dx^2}\frac{D(s)}{(B\rho)}}\frac{\Delta p}{p}$$

This term shows the effect of a sextupole, of length *l*, on the tune, Q, of the machine, as a function of Δp/p.

If we can make this term exactly balance the natural chromaticity then we will have solved our problem. But we have two chromaticities (ξ_h,ξ_v). However, the effect of a sextupole depends on β(s) and this varies around the circumference of the accelerator. So two types of sextupoles are used to correct the chromaticity of the accelerator. One is placed near QF quadrupoles, where β_h is large and β_v is small, this will have a large effect on ξ_h. Another placed near QD quadrupoles, where β_v is large and β_h is small, will correct ξ_v. Also sextupoles should be placed where D(s) is large, in order to increase their effect, since Δk is proportional to D(s).

Resonances

Now we go back to looking at individual particles and their betatron oscillations. For transverse betatron oscillations, a resonant condition is that: After a certain number of turns around the machine the phase advance per turn of the betatron oscillation is such that the oscillation exactly repeats itself. E.g. if the phase advance per turn = 120 degrees after 3 turns the betatron oscillation will exactly repeat itself. See below:

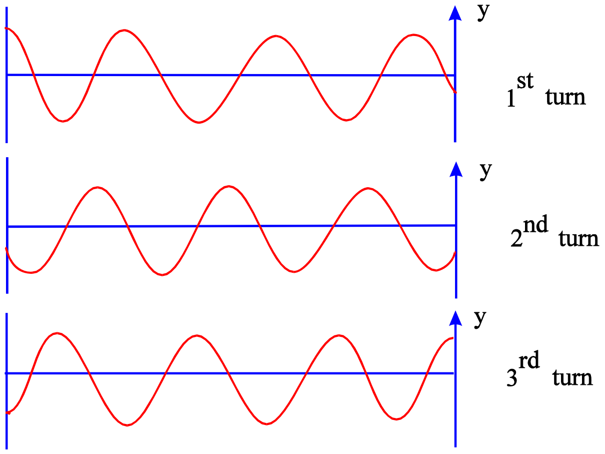


Figure 3: Third order resonant betatron oscillation for 3Q = 7

How does this look on our circular "normalised" phase space plot.

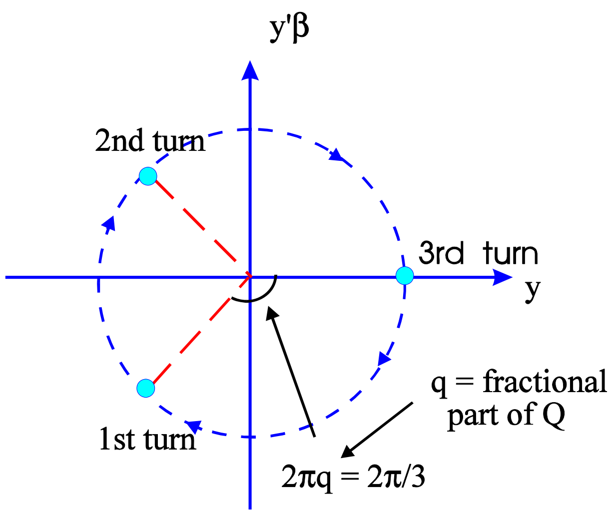
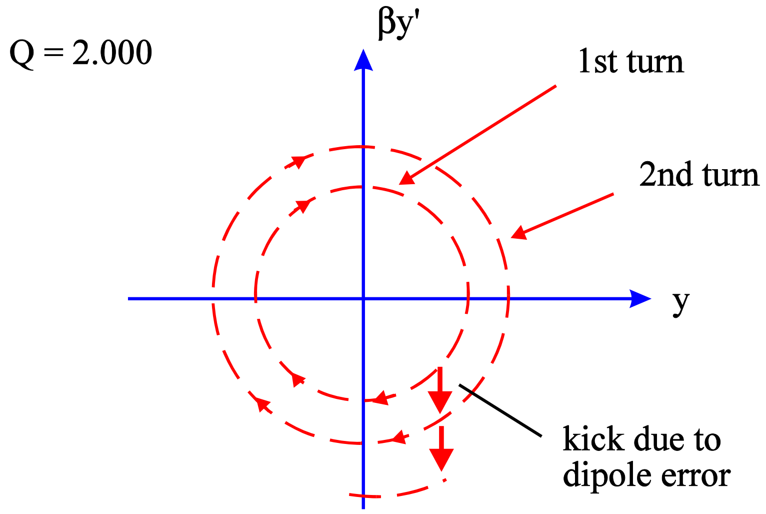
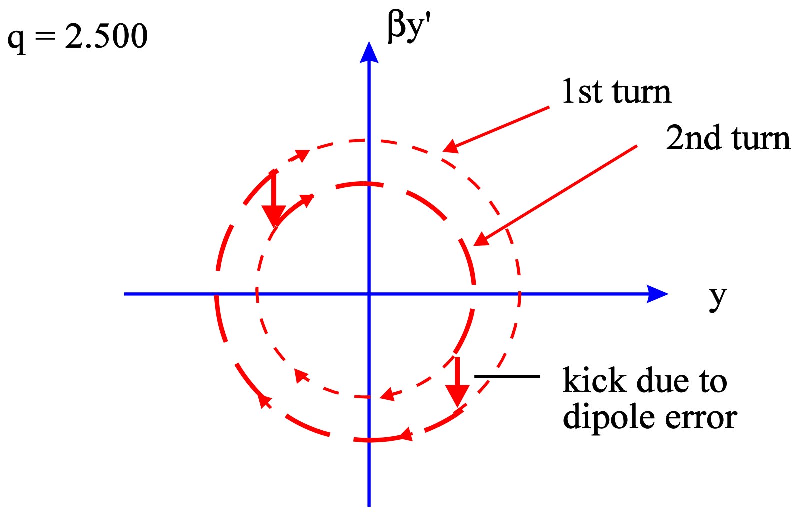


Figure 4: "Normalised" phase space plot for a third order resonant betatron oscillation

In figures 3 and 4, Q = 2.333, i.e. the 3rd order resonance, 3Q = 7
It is not possible to build a perfect machine, therefore we should ask, "What happens to our betatron oscillations due to different field errors?"

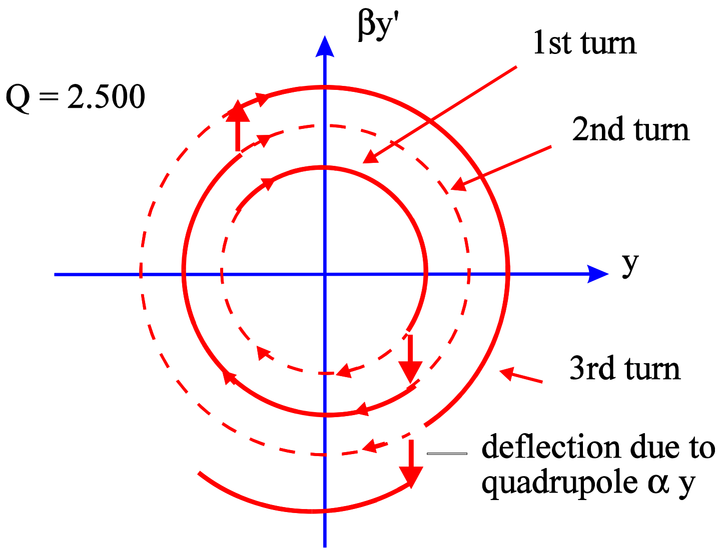
1) Dipole error, i.e. a constant deflection (independent of position). We will consider two cases Q = 2.000 and Q = 2.500.

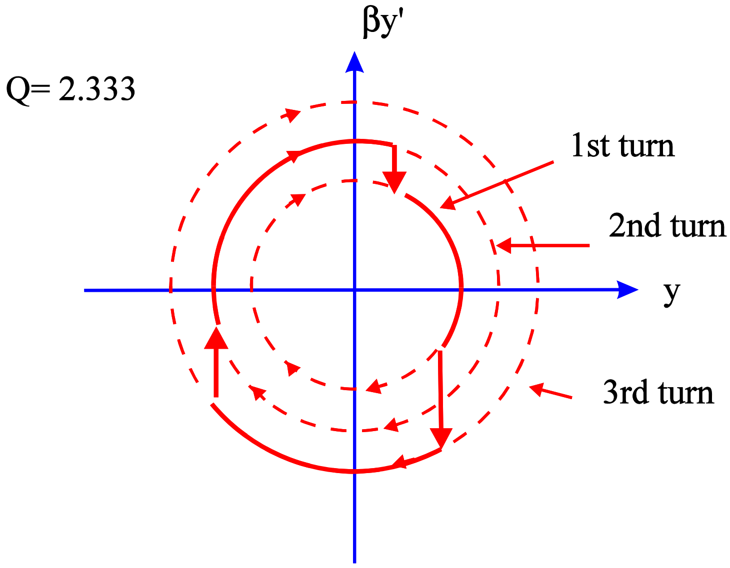




For Q = 2.0 the oscillation induced by the dipole error grows on each turn and the particle will eventually be lost as it’s oscillation amplitude grows. However, for Q = 2.5, the oscillation induced by the dipole error is canceled out every second turn, and the particle motion is stable. Therefore we conclude that for a dipole error we must avoid 1st order resonances but we can tolerate 2nd order resonances.

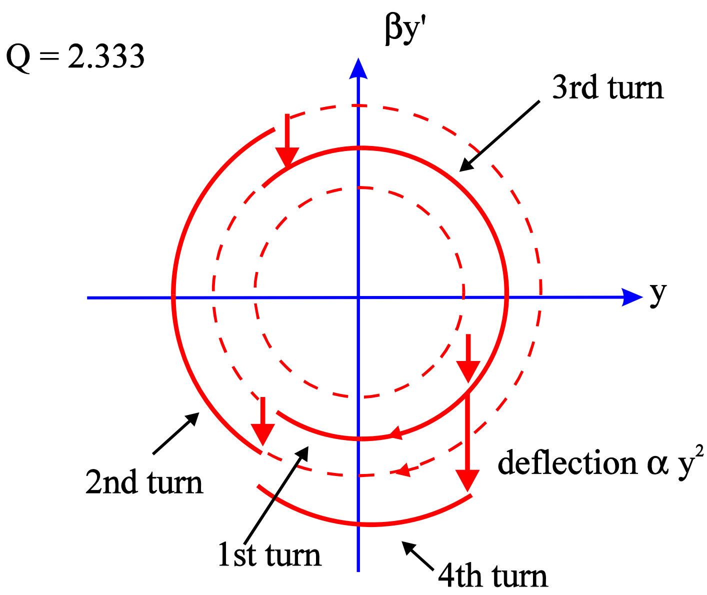
2) Quadrupole error, where deflection $\propto$ position. Again consider two cases, Q = 2.500 and Q = 2.333 i.e. second and third order resonances

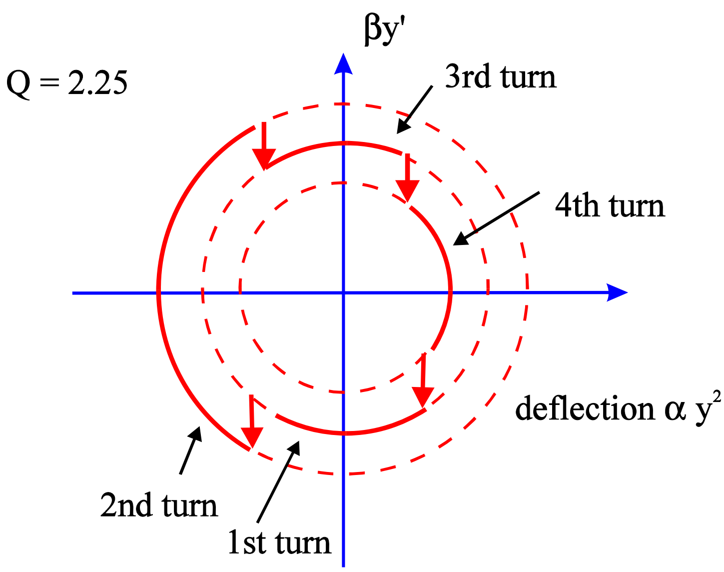




Here we see that for a second order resonances 2Q = 5 (Q = 2.500), the quadrupole error causes the amplitude to grow continuously, it is unstable. However, for Q = 2.333, the growth is canceled every 3 turns and is stable.

For a sextupole error the deflection $\propto$ (position)^2 . Again we consider two cases the third and fourth order resonances 3Q = 7 and 4Q = 9.



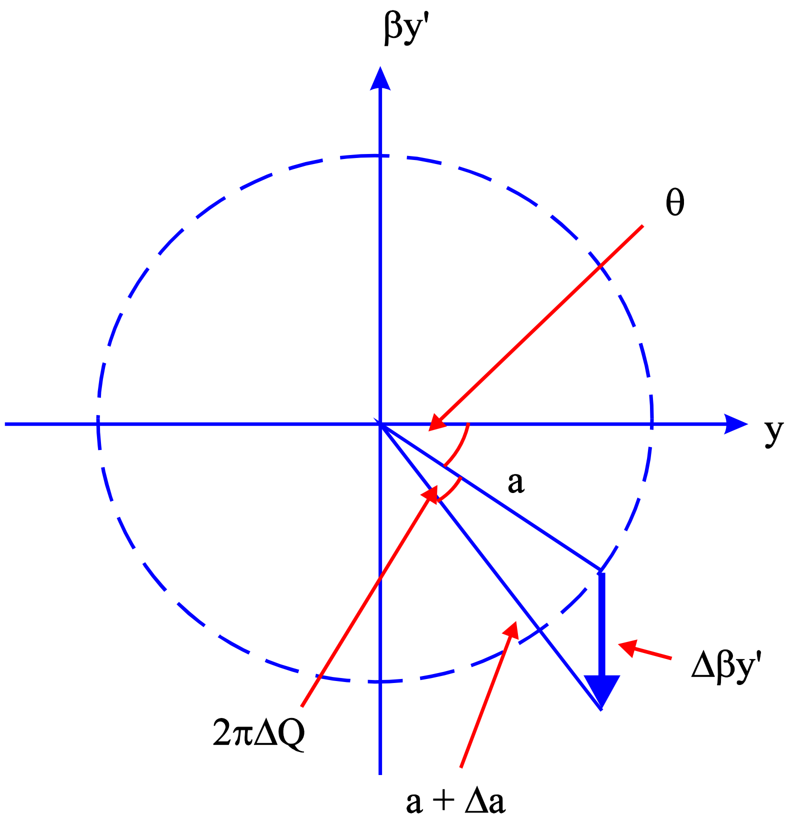


For 3Q = 7 the sextupole field increases the particle's amplitude, and therefore the particle is unstable again. For 4Q = 9, the motion is again stable every 4 turns.

Therefore it would appear that:-
Dipoles excite 1st order resonances
Quadrupoles excite 2nd order resonances
Sextupoles excite 3rd order resonances
This is rather unfortunate as we have seen that we have to include dipoles. quadrupoles and sextupoles in our accelerator. Therefore we also have to be very careful, when choosing the betatron tunes to avoid transverse resonant conditions, otherwise all out particles will have

continuously growing betatron amplitudes and will rapidly, after at most a few thousand turns, be lost on the accelerator vacuum chamber.
These resonant conditions have "growth times", i.e. the time it takes the betatron oscillation amplitude to grow and by some fixed amount. Higher order resonances have longer growth times, as it takes more turns to repeat the oscillation.

Now let us try to put some numbers in and treat this resonant growth in a slightly more rigorous manner. Below, the effect of some deflection, y', on the particle motion is plotted on a normalised phase space plot:-



Remember that the change in betatron phase after one turn, is $2\pi Q$, and the betatron phase at some point around the machine is θ. At this point apply a kick $\Delta(\beta y')$, which increases the oscillation amplitude by an amount Δa, i.e. the amplitude is increased from a to $(a+\Delta a)$, and the phase of the betatron oscillation changes by a small amount $(2\pi\Delta Q)$. Y does not change at the kick, and is given by:-

$$y = a\cos(\theta)$$

The small change in the phase advance $2\pi\Delta Q$ is given by:-

$$2\pi\Delta Q = \frac{\Delta(\beta y')\cos\theta}{a}$$

In a quadrupole $\Delta y' = lky$ (where k= normalised gradient and l = length)

$$\Delta Q = \frac{1}{2\pi}\frac{\beta\cos(\theta)\ \ell ak\cos(\theta)}{a}$$

$$\Delta Q = \frac{1}{2\pi}\ell\beta k\cos^2(\theta) = \frac{1}{4\pi}\ell\beta k(\cos(2\theta)+1)$$

This is valid for one turn, on the second turn the phase (θ) will have changed by 2πQ. On the nth turn θ = θ + 2nπQ. Therefor over several turns:-

$$\Delta Q = \frac{1}{4\pi}\ell\beta k\left[\sum_{n=1}^{\infty}\cos\big(2(\theta+2\pi nQ)\big)+1\right]$$

If we average over many turns, we obtain:-

$$\Delta Q = \frac{1}{4\pi}\beta kds.$$

Since Σcos θ → 0. This is exactly the result we obtained in chapter 4 for the change in Q due to a quadrupole. (Note that Q does change slightly on each turn). At the moment we are interested in the change in oscillation amplitude:-

$$\Delta a = \beta\Delta y'\sin(\theta) = l\beta\sin(\theta)\ a\ k\cos(\theta)$$

$$\therefore \frac{\Delta a}{a} = \frac{\ell\beta k}{2}\sin(2\theta)$$

Over many turns we obtain:-

$$\frac{\Delta a}{a} = \frac{\ell\beta k}{2}\sum_{n=1}^{\infty}\sin\big(2(\theta+2n\pi Q)\big)$$

Again Δa→0, since Σsin(θ) → 0. Therefore, there is no change in overall oscillation amplitude. This argument is valid unless q = 0.5 (q = fractional part of Q), now the phase term, 2(θ + 2nπQ) is constant and, therefore:-

$$\sum_{n=1}^{\infty}\sin\big(2(\theta+2n\pi Q)\big) = \infty \quad \text{and} \quad \frac{\Delta a}{a} = \infty.$$

In this case the amplitude will grow continuously until the particle is lost. Therefore we must not set q = 0.5. However Q has a range of values:-

$$\Delta Q = \frac{\ell\beta k}{2\pi}$$

Where β = β value at the quadrupole.

So even if q is not exactly 0.5, it must not be too close, or at some point it will find itself at exactly 0.5 and "lock on" to the resonant condition. This width, $\Delta Q = \frac{\ell\beta k}{2\pi}$, is called the <u>stopband</u> of the resonance and means that we must not only avoid resonances but also not approach them too closely!

We can repeat this exercise for a sextupole error.

$$\Delta y' = \ell k y^2$$

$$\Delta Q = \frac{1}{2\pi}\ell\beta k a \cos^3\theta$$

$$\Delta Q = \frac{1}{8\pi}\ell\beta k a\left(\cos 3\theta + 3\cos\theta\right)$$

Similarly:-

$$\frac{\Delta a}{a} = \ell\beta k a \sin\theta\cos^2\theta \quad = \frac{\ell\beta k a}{2}\left[\cos 3\theta + \cos\theta\right]$$

Summing over many turns exactly as before we obtain:-

$$\frac{\Delta a}{a} = \frac{\ell\beta k a}{2}\sum_{n=1}^{\infty}\cos 3(\theta + 2\pi n Q) + \cos(\theta + 2\pi n Q)$$

Therefore $\Delta a/a$ is zero unless Q =1/3, third order resonance, or Q is an integer, first order resonance. This confirms that sextupoles excite first and third order resonances. Also notice that the width of the stop-band, ΔQ, now depends on the particle's oscillation amplitude "a".

Similarly, for an octupolar error.

$$\Delta y' = \ell k y^3$$

$$\therefore \Delta Q = \frac{1}{2\pi}\ell\beta k a^2 \cos^4 Q\theta$$

This leads to an amplitude, Δa, dependence $\propto a^2(\cos 4\theta + \cos 2\theta)$

where a^2 = betatron oscillation amplitude squared

$\cos 4\theta$ = Fourth order resonance term

$\cos 2\theta$ = Second order resonance term

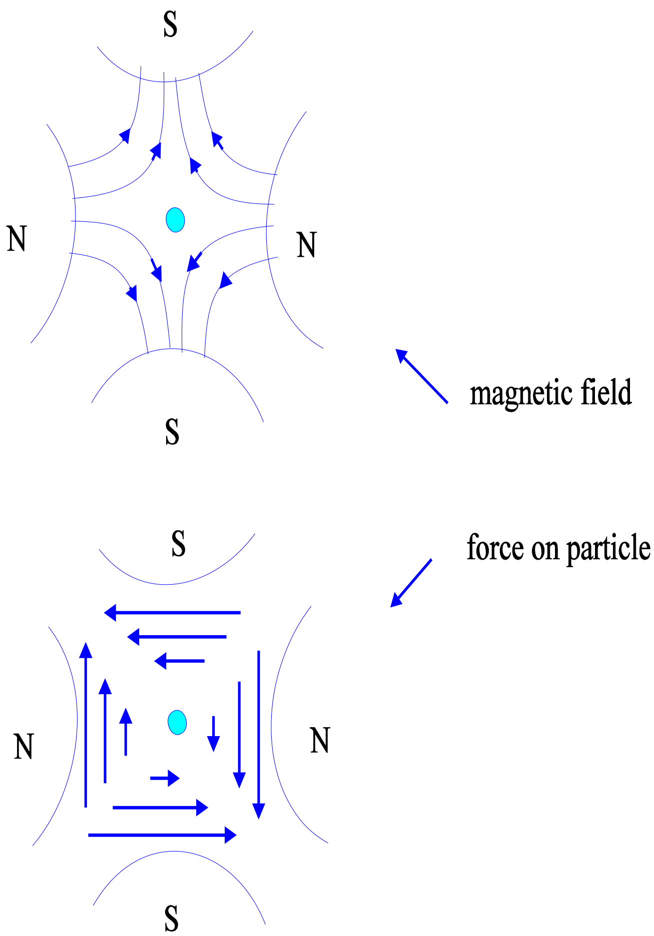
This means that even small octupolar errors are very important for larger amplitude betatron oscillations, because the stopband width increases as the amplitude squared.

Coupling

This is a phenomena, which converts horizontal betatron motion into vertical motion and vice versa (like a pair of coupled pendulae or is it pendulums?)

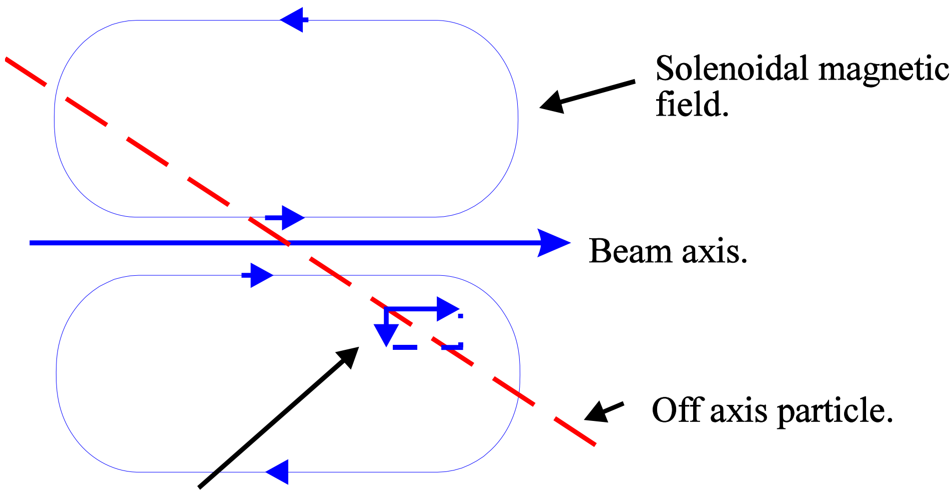
Fields, which will excite coupling

1) Skew quadrupole, This just a normal quadrupole, which has been rotated by 45° about it's longitudinal axis.



This 45° rotation has the effect that a horizontal displacement now induces a vertical deflection, and a vertical displacement induces a horizontal deflection. Therefore this skewed quadrupole no longer focuses (or defocuses) the beam, but transfers energy from one transverse plane to the other. Any quadrupole, which is not perfectly aligned, will have a skew component and will excite coupling.

Another way to excite coupling is a solenoid (or longitudinal field)

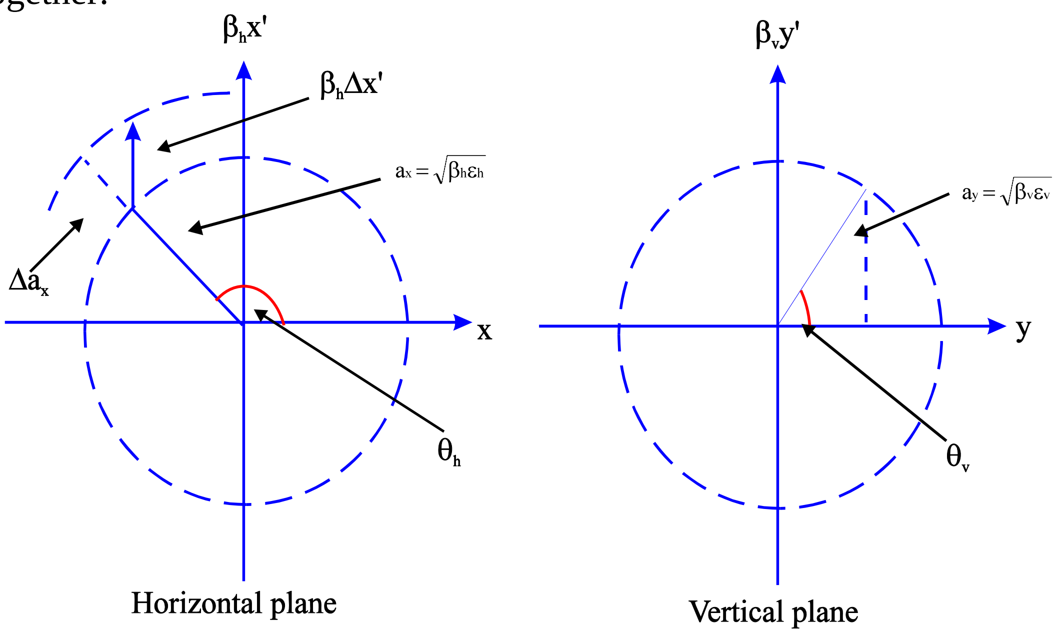


If a particle travels exactly parallel to the magnetic field lines, there is no magnetic force ($\underline{V} \wedge \underline{B}$). But the particles all execute betatron oscillations, and so there is always some transverse velocity component. This means that there is a deflection, in this case out of the page. In this way the vertical transverse velocity component has caused a horizontal deflection, and again the transverse motions are coupled.

Let us put a single fully skewed quadrupole, of length *l*, in our accelerator, with:-

$$k = \frac{1}{(B\rho)} \frac{dB_x}{dx}$$

Now we must consider both horizontal and vertical phase space diagrams together.



There will be a horizontal deflection, $\Delta x'$, at the skew quadrupole due to the vertical displacement, y, $y = \sqrt{\beta_V \varepsilon_V}\cos(\theta_v)$.:-

$$\text{where } \Delta x' = yk\ell = \sqrt{\beta_V \varepsilon_V}\, k\ell \cos(\theta_v).$$

Now we can calculate the change in horizontal amplitude Δa_x:-

$$\Delta a_x = \beta_H \sqrt{\beta_V \varepsilon_V}\, k\ell \cos(\theta_v)\sin(\theta_h)$$

$$\therefore \frac{\Delta a_x}{a_x} = \sqrt{\frac{\varepsilon_V}{\varepsilon_H}}\sqrt{\beta_H \beta_V}\, k\ell \cos(\theta_v)\sin(\theta_h)$$

$$\therefore \frac{\Delta a_x}{a_x} = \sqrt{\frac{\varepsilon_V}{\varepsilon_H}}\sqrt{\beta_H \beta_V}\, \frac{k\ell}{2}(\sin(\theta_v - \theta_h) + \sin(\theta_v + \theta_h))$$

Again summing over n turns:-

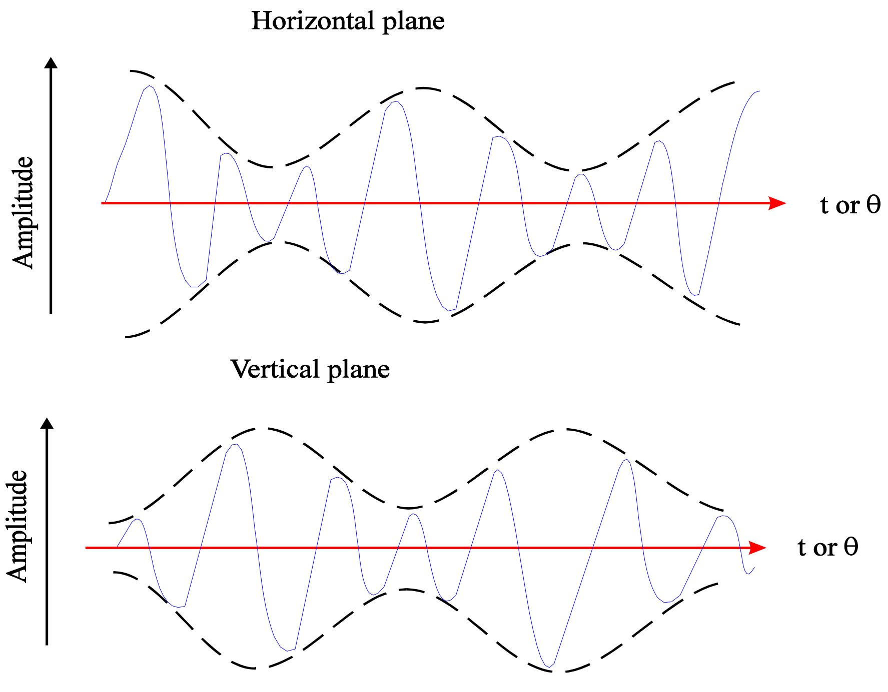
$$\frac{\Delta a_x}{a_x} = \sqrt{\frac{\varepsilon_V}{\varepsilon_H}}\sqrt{\beta_H \beta_V}\, \frac{k\ell}{2}\sum_{n=1}^{\infty}\left[\sin(\theta_- + 2\pi n(Q_h - Q_v)) + \sin(\theta_+ + 2\pi n(Q_h + Q_v))\right]$$

Where $\theta_+ = \theta_h + \theta_v$, and $\theta_- = \theta_h - \theta_v$.

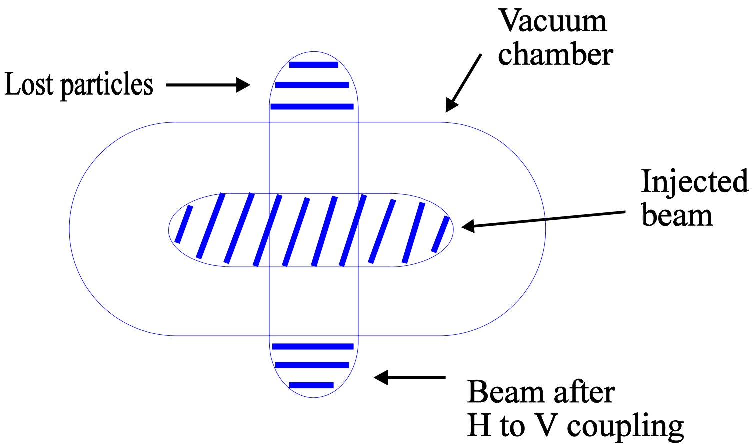
Similarly for the vertical plane:-

$$\frac{\Delta a_y}{a_y} = -\sqrt{\frac{\varepsilon_H}{\varepsilon_V}}\sqrt{\beta_H \beta_V}\, \frac{k\ell}{2}\sum_{n=1}^{\infty}\left[\sin(\theta_- + 2\pi n(Q_v - Q_h)) + \sin(\theta_+ + 2\pi n(Q_h + Q_v))\right]$$

By summing these two coupled expressions over many turns, n, we see that the amplitudes of our betatron oscillations will oscillate back and forth, between the horizontal and vertical planes.



The speed at which this transfer takes place depends on both $(Q_h - Q_v)$ and $(Q_h + Q_v)$. The result is a beating between horizontal and vertical oscillations. Remember this is not β changing but ε_V and ε_H, i.e. the amplitudes of particles' betatron oscillations, and therefore coupling is dangerous, as it changes temporarily the transverse beam emittances. We normally expect $\varepsilon_V<\varepsilon_H$, and so we build elliptical vacuum chambers for particle accelerators. But if we inject a beam with $\varepsilon_V<\varepsilon_H$ into a machine with strong skew quadrupole component, after a few turns $\varepsilon_V>\varepsilon_H$ and most of the beam will be lost.



But we have also seen that the coupling itself depends on the values of Q_h and Q_v. So what happens to $\frac{\Delta a}{a}$ if we let $Q_h \pm Q_v$ = integer. This condition is called a coupling resonance.
Remember that:-

$$\frac{\Delta a_x}{a_x} = \sqrt{\frac{\varepsilon_V}{\varepsilon_H}}\sqrt{\beta_H\beta_V}\,\frac{k\ell}{2}\sum_{n=1}^{\infty}\left[\sin(\theta_- + 2\pi n(Q_h - Q_v)) + \sin(\theta_+ + 2\pi n(Q_h + Q_v))\right]$$

If $Q_h \pm Q_v$ = integer then one of the two sin terms above is constant and non-zero. In this case, instead of $\Delta a/a$ oscillating back and forth, $\Delta a/a\to\infty$ as $n\to\infty$. In this case the particles will all be lost.
So we must minimise coupling by aligning all the machine quadrupoles as well as we can. We can also include skew quadrupoles to compensate unwanted coupling. For a Collider, this must be done for any solenoid experimental magnets in the machine. Usually two skew quadrupoles are required, one at high β_h and one at high β_v (exactly as for sextupoles) in order to fully compensate the coupling in both planes. Additionally, we should also avoid tuning Q_h and Q_v too close to coupling resonances.

Chapter 6 Longitudinal motion and RF

In this chapter we will look at motion in the longitudinal plane. (S or θ) let's start with some words about revolution frequency and how it changes with momentum. At constant magnetic field:

$$\frac{\Delta f}{f} = \frac{\Delta v}{v} - \frac{\Delta r}{r}$$

where Δv/v = change in velocity, and Δr/r = change in orbit length.

If the particle's momentum increases, then initially the particle travels faster, but in addition the particle will follow a longer orbit, as it now has a higher momentum and this second effect will tend to reduce the revolution frequency.

Remember $\frac{\Delta r}{r} = \alpha p \frac{\Delta p}{p}$, where αp = momentum compaction factor.

$$\therefore \frac{\Delta f}{f} = \frac{\Delta v}{v} - \alpha p \frac{\Delta p}{p}$$

But $\frac{\Delta v}{v} = \frac{\Delta \beta}{\beta}$ $\left(\beta = \frac{v}{c}\right)$ and relativity tells us that

$$p = \frac{E_0 \beta \gamma}{c} \quad \text{where } \gamma = \left(1 - \beta^2\right)^{\frac{-1}{2}}$$

$$\therefore \frac{dp}{d\beta} = \frac{E_0}{c} \gamma^3$$

$$\therefore \frac{dp}{p} = \gamma^2 \frac{d\beta}{\beta}$$

$$\therefore \frac{\Delta f}{f} = \left(\frac{1}{\gamma^2} - \alpha p\right) \frac{\Delta p}{p}$$

αp is fixed by the lattice of the accelerator i.e. by the layout and magnetic field of the quadrupoles, but $1/\gamma^2$ varies as the momentum increases $\left(E = \gamma E_0\right)$.

At low energy: $\beta \ll 1 \quad \gamma \to 1 \quad \frac{1}{\gamma^2} > \alpha p$

∴ The revolution frequency increases as the particle's momentum increases.

At high energy: $\beta \approx 1 \quad \gamma \to$ very large $\quad \therefore \frac{1}{\gamma^2} \langle \alpha p$

$\therefore$ The revolution frequency decreases as the momentum increases.

This is reasonable if we consider that as we accelerate a particle, at low momentum it's increasing velocity is very important compared with the increase in momentum. However, as the velocity approaches the velocity of light, the change in velocity becomes very small and the increasing momentum is dominant. Figure 1 shows the variation of revolution frequency with particle momentum.

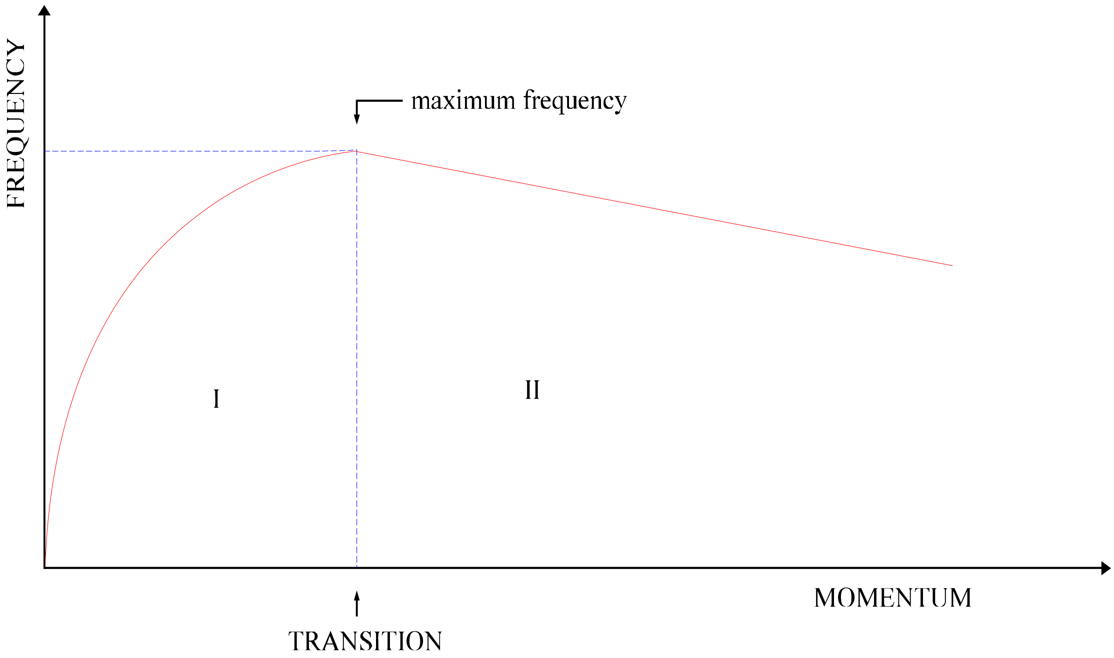


Figure 1: Revolution frequency .v. particle momentum

In region I in figure 1, $\beta \langle\langle 1$, and the particle velocity increases rapidly with momentum leading to a rapid increase in revolution frequency. At **"Transition"**, then $\frac{1}{\gamma^2} = \alpha$, and the revolution frequency reaches a maximum value. Finally in region II, $\beta \approx 1$, so the particle velocity is essentially constant and the revolution frequency decreases, as a result of the increase in beam momentum.

Rewrite our equation for the revolution frequency:

$$\frac{\Delta f}{f} = \overbrace{\left(\frac{1}{\gamma^2} - \frac{1}{\gamma_{TR}^2}\right)}^{\eta} \frac{\Delta p}{p} \qquad \text{and define } \gamma_{TR}^2 = 1/\alpha p$$

η now gives us the relationship between revolution frequency and momentum for a given accelerator, but note that η is both lattice and momentum dependent.

When the particle momentum is equal to the transition momentum, i.e. $\gamma = \gamma_{TR}$ $\frac{\Delta f}{f} = 0$ then all particles will have the same revolution frequency. We will see later that transition is very important in proton machines, as normally it lies within the range of operating momenta of the machine, e.g. transition occurs at around 6 GeV/c in CERN PS. Conversely it does not exist for electron machines, since for leptons $\beta = 1$. So while most proton machines must operate above and below transition and cross through the transition momentum at some point, lepton rings operate exclusively above transition.

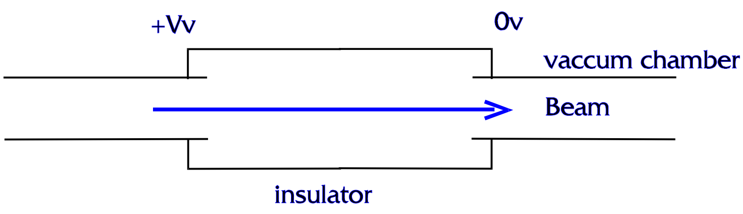
RF SYSTEM

The Radio Frequency (RF) system is used:-

1) Proton (and let's not forget...antiproton) machines
   a) To accelerate/decelerate beam.
   b) To make beam measurements
   c) To increase luminosity in Colliders...

2) Lepton machines:
   a) To accelerate beam.
   b) To compensate for energy loss due to emission of synchrotron radiation.

In order to accelerate the beam of particles we need a longitudinal electric field. Magnetic fields cause deflections to the particle trajectory but they do **not** change the overall particle momentum. So we must generate a longitudinal voltage, which is applied across an isolated gap in the vacuum chamber.



If we use a DC voltage, over a full turn, we get no overall acceleration, as the particle will be accelerated through the gap (+V), and decelerated over rest of the circumference (-V). Therefore, we use an oscillating voltage, so that the particle sees an accelerating voltage at the gap, and the voltage then cancels out as the particle goes around the rest of the machine. However we must make sure that the particle always sees an accelerating voltage at the gap, so the RF frequency must always be an integer multiple of the revolution frequency, which depends on the particle's revolution frequency and hence it's momentum.

$$h(\text{integer}) = \text{harmonic number} = \frac{RF\ frequency}{rev.frequency} = \frac{f_{RF}}{f_{rev}}$$

For leptons we use a fixed frequency as β=1, but for low energy protons we need a variable frequency as β<1. This frequency will vary directly with the β of the particles, therefore it has to be derived from the particle momentum.

In order to get a large accelerating voltage we put a resonating cavity around the gap. Which resonates at the chosen RF frequency. How does a particle in our machine react to this voltage? Let us start with a machine above transition, therefore it is important to recall that: Higher energy particles will have a longer orbit and a lower revolution frequency, which will delay their arrival at the accelerating cavity. Conversely lower energy particles will have a shorter orbit, a higher revolution frequency, and will arrive earlier at the accelerating cavity. We will go back to a machine below transition later.

Imagine two particles in our accelerator, particle A, which has a momentum (or energy), which corresponds exactly to the RF frequency (h = 1). Suppose that when particle A passes through the RF cavity, the voltage is zero. In this case every time A passes through the cavity it will see zero voltage, as it's revolution frequency is the same as the RF frequency. Particle A is synchronous with the RF voltage. The second particle B, initially, arrives at the cavity at the same time as A, but it's momentum is slightly higher than A's, and therefore it's revolution frequency is slightly lower. See figure 2.

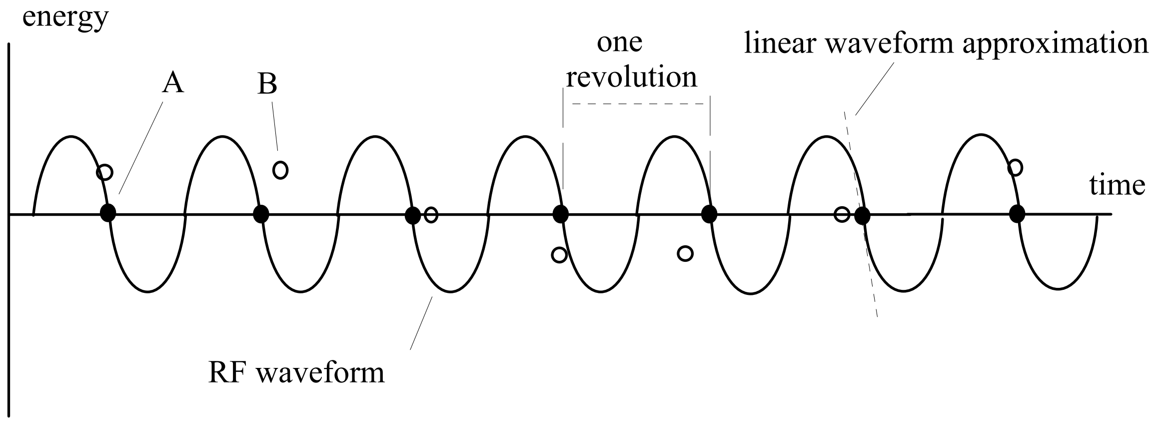


Figure 2: Shows the energy variation of two particles over several turns. Particles above the centre line have a higher energy than those below. Accelerating RF voltage is also above then central line.

In figure 2, particle A always passes through the cavity when there is no RF voltage (either accelerating or decelerating). However, particle B is not so lucky... On the first turn B arrives at the same time as A but with a higher energy, therefore on the second turn it arrives later than A and sees a decelerating RF voltage, which reduces its energy to exactly that of A. Now on the third turn it still arrives later than A as is has exactly the same energy/frequency, therefore B is decelerated still more and now has a lower energy than A. On the fourth turn B now arrives at the same time as A, as its energy is lower and therefore its revolution frequency is higher, so B sees no acceleration or declaration and is still at a lower energy than A. On the fifth turn B now arrives before A and sees an accelerating voltage, which means it now has the same energy and revolution frequency as A again. Therefore on the sixth turn B still arrives before A and is accelerated again. Now B has a higher energy and a lower revolution frequency than A. So on the seventh turn B now arrives at the same time as A but with a higher energy. This is just the situation that we had at the beginning.

Particle A is called the Synchronous particle, as it is exactly synchronised with the RF frequency. All the other particles in the accelerator, like particle B, will oscillate longitudinally around A under the influence of the RF system. These oscillations are called synchrotron oscillations. This longitudinal motion is plotted in longitudinal phase space. See Figure 3

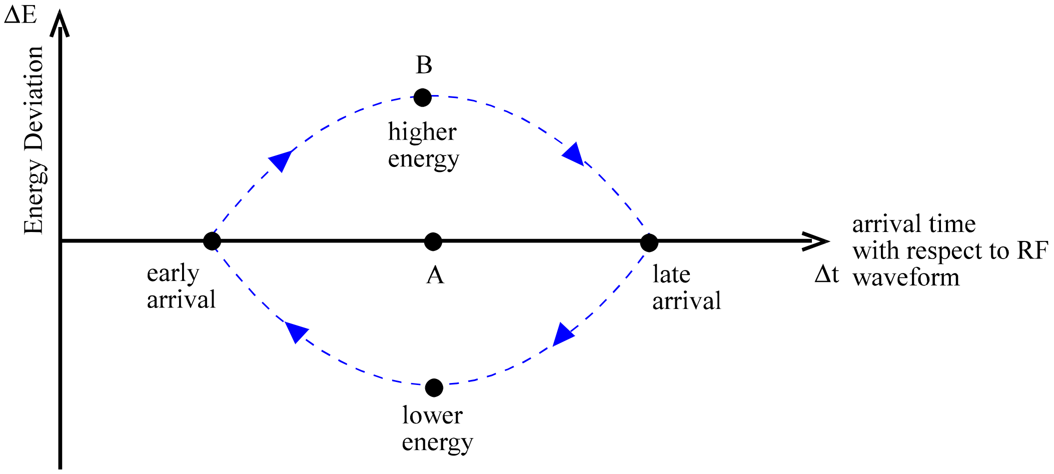


Figure 3: Longitudinal phase space plot for particles A and B from Figure 2

Thus all the particles in the accelerator rotate around the synchronous particle on the longitudinal phase space plot. This means that instead of being spread uniformly around the circumference of the accelerator the particles get “clumped” around the synchronous particle in a bunch. This bunch is contained in an RF bucket. For small energy deviations the particles follow a circular path inside the bunch. For larger energy deviations these circles get flattened into ellipses, as the restoring force drops off from the linear central valves. Obviously the form of these ellipses depends on the size of the restoring force, which is determined by the RF Voltage on the cavity (and the RF harmonic number). See Figure 4.

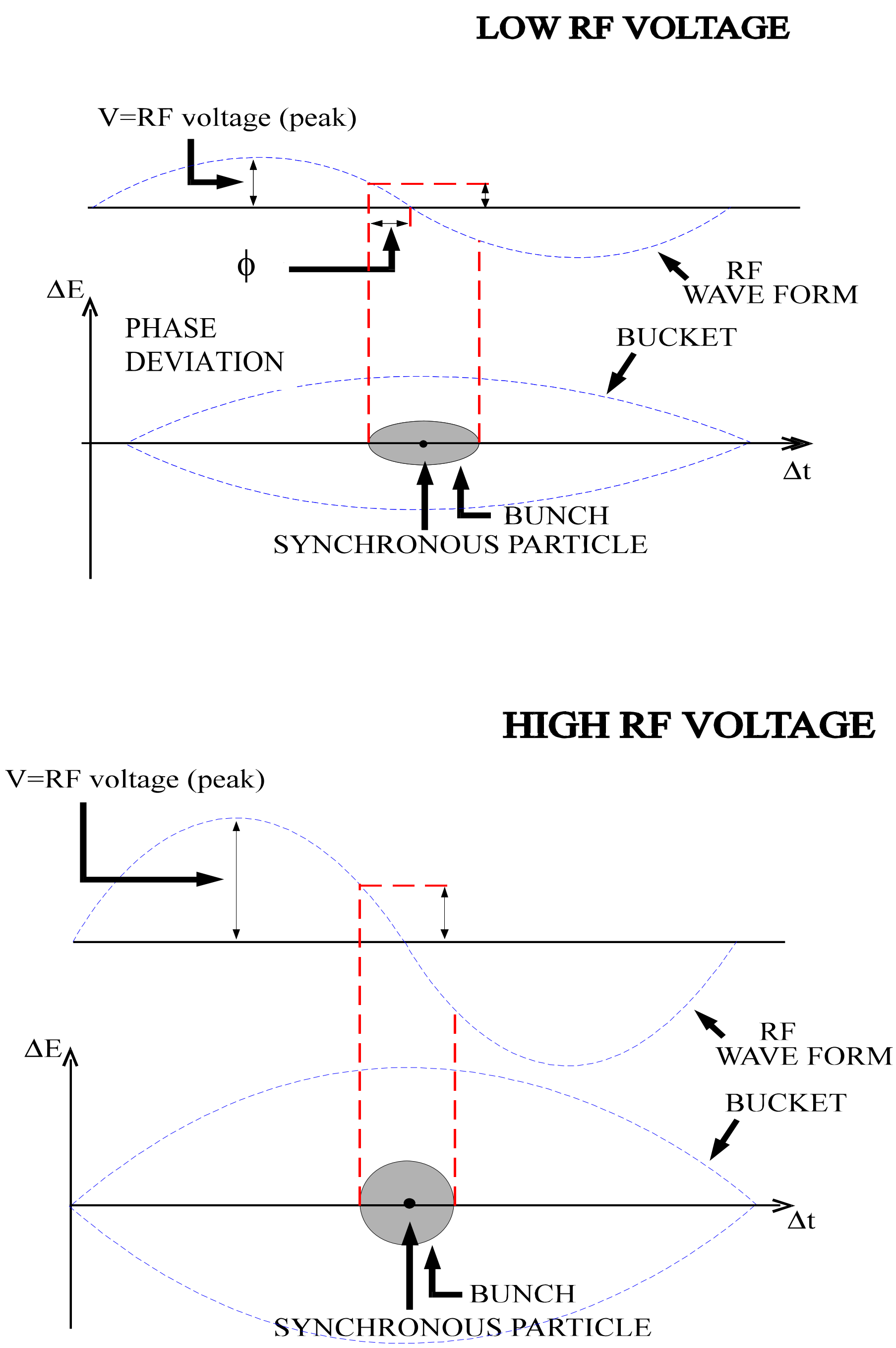


Figure 4: Bunch shapes for different RF voltages

For a given set of RF parameters (voltage and harmonic number), there will some maximum energy deviation, which can be trapped around the synchronous particle. The trajectory of this particle in longitudinal phase space defines the size and the form of the RF bucket. See Figure 5. This bucket can also be thought of as a potential well inside which the particles remain trapped and oscillate back and forth provided their energy deviation is less than the depth of the bucket. See Figure 5.

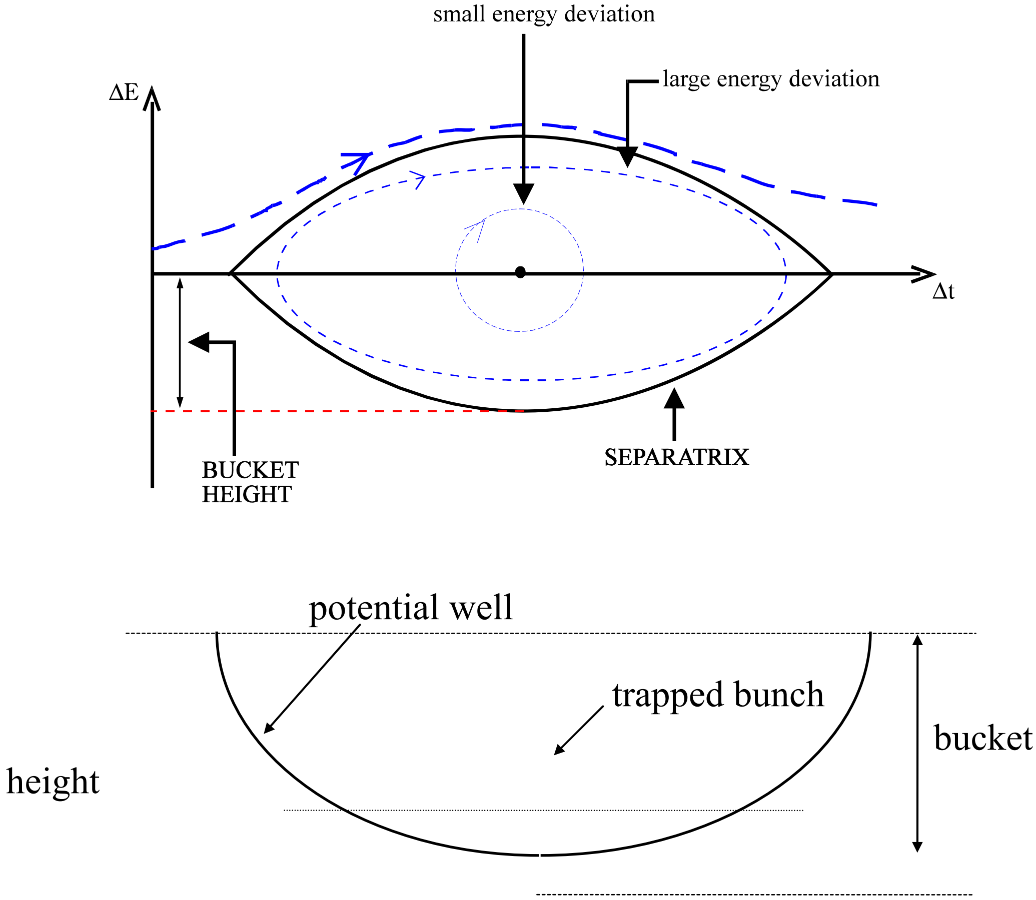


Figure 5: An RF bucket

In Figure 5 particles with an energy deviation greater than bucket height cannot be trapped and fall outside the RF Bucket. Any particles inside form the bunch. The RF bucket is often called a Separatrix. These buckets will exist as soon as the RF is put on and the number of buckets = h = harmonic number = f_{rf}/f_{rev}. We can fill as many of these buckets with particles as we wish.

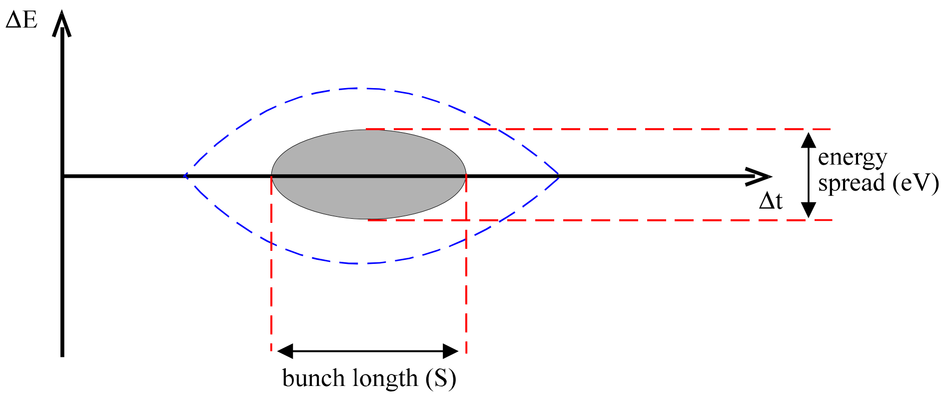


Figure 6: Definition of longitudinal emittance

We define a longitudinal emittance as an area of longitudinal phase space, exactly as we did for transverse emittance, see figure 6. The area of bunch = longitudinal emittance (units (eVs)), and we also express the bucket area in the same way, as a longitudinal acceptance of the bucket (the units are again (eVs)).

We have based all our discussions so far on the definition of the synchronous particle as the particle which sees no RF voltage, but for electrons, for example, we must include synchrotron radiation energy loss. Suppose that the electron loses an energy Vs/turn. See figure 7.

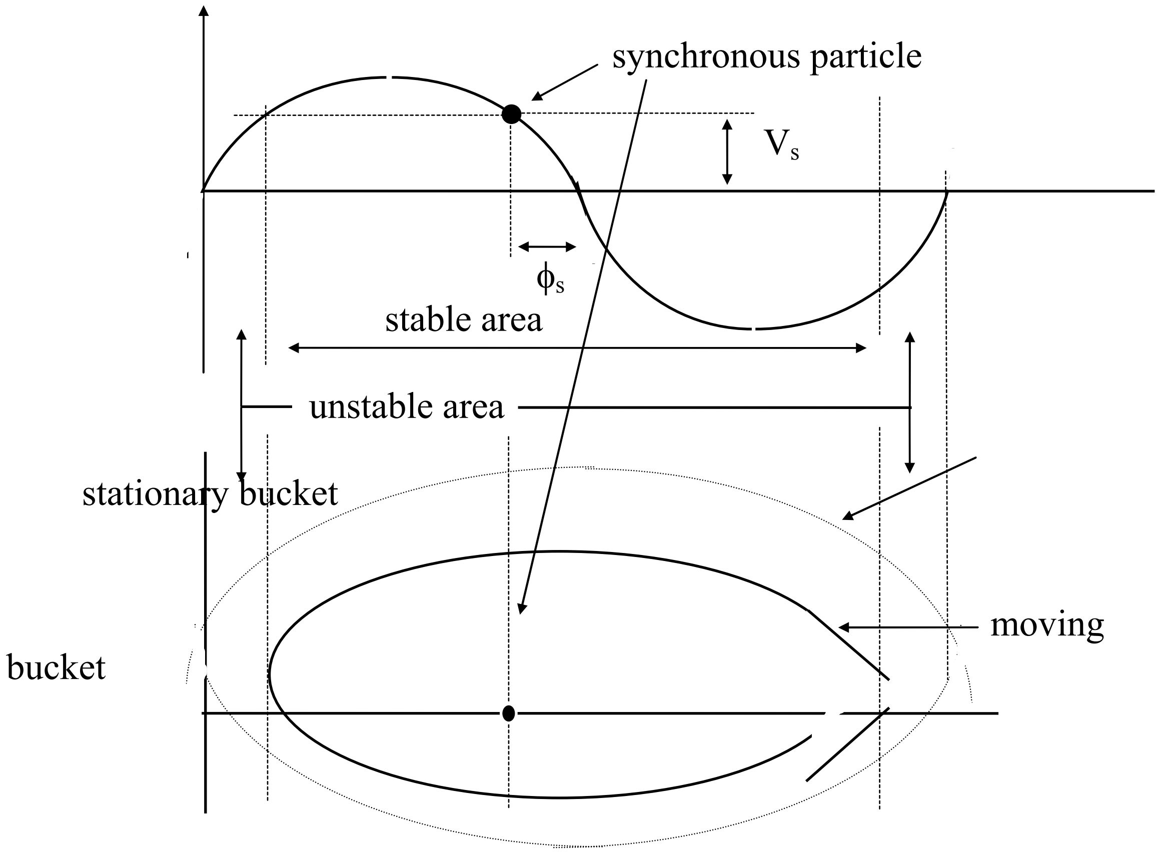
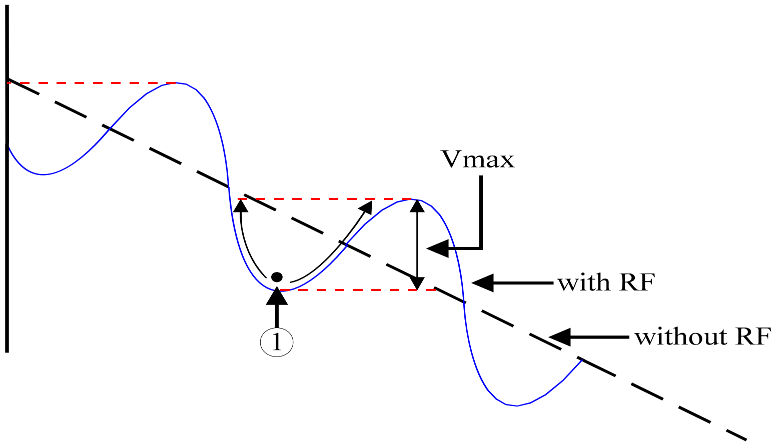


Figure 7: The synchronous particle and the RF bucket including energy loss.

The synchronous particle losses energy V_s on one turn therefore it no longer sits at the zero crossing of the wave form but at some phase ϕ_S (synchronous phase).

So that $V\sin\phi_s = V_s$ = energy loss/turn. V = RF peak voltage, and $\sin\phi_s$ is often written as ☝. Any particle executing synchrotron oscillations around this new synchronous particle will soon find it self at a lower energy than the synchronous particle. Now it must gain at least V_s on each turn from the RF system, or it will never regain enough energy to catch the synchronous particle again, and hence will be lost from the bunch. This defines an "unstable area", which is no longer inside the RF bucket in figure 7. The original RF bucket we defined, without the effect of energy loss/turn is called a "stationary bucket". The new bucket, including the energy loss/turn is called a "moving bucket". For the same RF voltage the moving bucket is always smaller than the stationary bucket.

We could also imagine this "energy loss/turn" as a slope put below the potential well representation of the RF bucket in figure 6. The particles would then slide down this slope to lower and lower energies if the RF did not give back the lost energy on each turn. See figure 8.



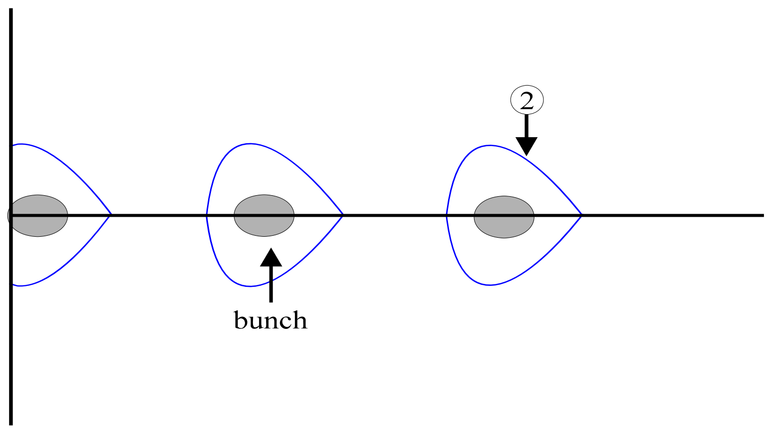


Figure 8: RF buckets for an accelerating RF system

The particles only remain trapped (① in the first part of figure 8) if their maximum energy excursion is less than V_{max}. This changes the shape of our RF buckets (② in the second half of figure 8). Any particle, which

finds itself between the stable buckets will slide down the potential and continuously lose energy until it is lost out of the accelerator.
Note also how the bucket area has been reduced. If we increase the energy loss/turn still further, i.e. increase the slope of the potential, the bucket area gets smaller, but the bucket area will increase if we increase RF voltage.

As well as restoring the energy lost due to synchrotron radiation loss, the RF system can supply extra longitudinal energy to accelerate the particles. In this case we increase the magnetic field smoothly with the RF system on. Now, due to the increase the magnetic field, the particles follow a shorter path, and hence arrive earlier at the RF cavity, which looks exactly as if they have lost energy. Therefore the RF system compensates for this apparent "energy loss" to put the particles back at the correct revolution frequency and thus accelerates the bunch or bunches.

If figure 8 we can see that the bucket area depends directly on the rate of acceleration. Faster acceleration, i.e. ramping the magnets faster, looks like an increase in the apparent energy loss, or an increase in the overall slope we have put on the potential well representation of the RF buckets. This reduces the bucket area. Therefore faster acceleration needs a higher RF voltage. But the RF voltage only determines the bucket area and not the rate of acceleration. That is determined solely by the rate at which the magnetic field is increased, i.e. the rate at which the magnets are ramped. However, the RF voltage must be high enough to ensure that the bucket area is always much greater than bunch area, and hence avoid the possibility of losing particles out of bucket during the acceleration. Since once a particle has dropped out of the bucket during acceleration it will soon be lost on the accelerator vacuum chamber.

Finally let us go back, as promised, to the case of a machine operating below transition, i.e. a low energy proton accelerator. Now the revolution frequency increases as energy increases. Therefore higher energy particles arrive earlier not later. This means that the direction of the particle rotation around the longitudinal phase space plot is reversed. See figure 9.

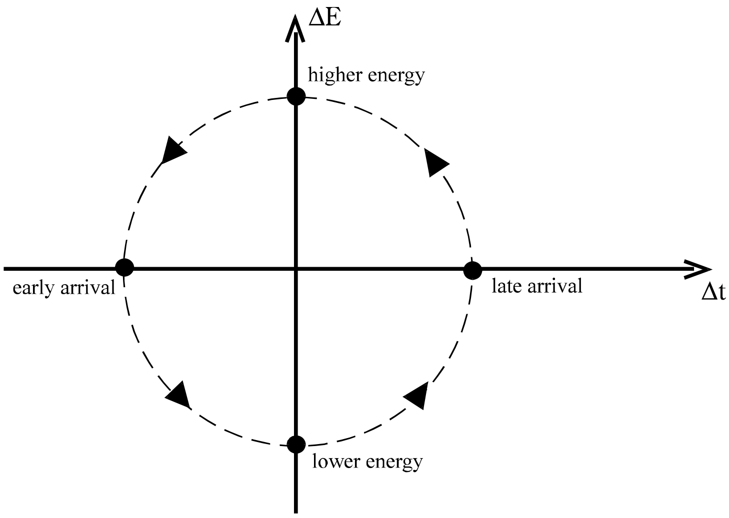


Figure 9: Longitudinal phase space below transition

But a more interesting and important question is what happens to the particles in the bunch with respect to the RF waveform?
Above transition we saw that the higher energy particles arrive later at the cavity and therefore must see a decelerating voltage to keep them oscillating around the synchronous particle. This determines the sign of the slope of the RF waveform at the synchronous particle. The synchronous particle must sit on the negative slope of the waveform, above transition. However, below transition, the higher energy particle will arrive sooner, not later, but it still needs to see a decelerating voltage. Therefore the synchronous particle must sit on the positive slope of the RF waveform below transition. See Figure 10.
At transition, the phase of the RF waveform must jump by exactly 180°, in order to keep bunches on the stable synchronous point in the accelerating bucket. This is a very important point and a non-negligible complication for the PS RF system! During every proton acceleration cycle in the PS, the bunched beam must cross transition, as the transition occurs at 6 GeV, with injection at 1.4 GeV and ejection at 26 GeV.

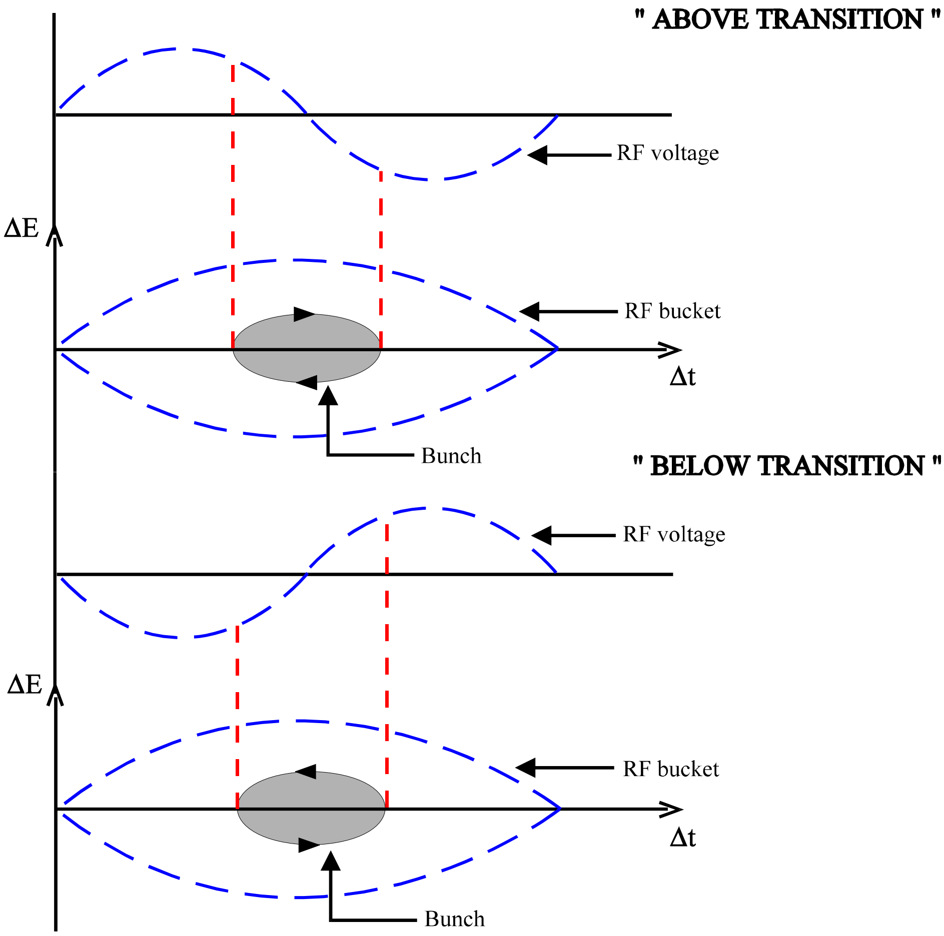


Figure 10: RF buckets above and below transition

Chapter 7: What is special about electrons?

In this chapter we will discuss the effect of the emission of **Synchrotron Radiation** on the behaviour of particles in our accelerator/storage ring. We will cover:

1) Rate of energy loss.
2) Longitudinal damping.
3) Transverse damping.
4) Quantum fluctuations.

An accelerating charge emits photons in the form of electromagnetic radiation. This is how any antenna works, an oscillating current generates E-M waves. However the emitted power depends very strongly on the mass of the particle and it's momentum. Therefore we only worry about the effects of synchrotron radiation for leptons and very high energy protons. Here we will only consider leptons.

We have two accelerating system in our particle accelerator, one longitudinal, the RF system, and a second in the two transverse planes, the magnetic guide fields, including both dipole and quadrupole magnets.

For an accelerating force parallel to an electron's velocity, the rate at which a relativistic electron radiates electromagnetic energy is proportional to the square of the accelerating force. This rate increases if there is an angle between the accelerating force and the electron's velocity, and if this angle is 90° this increase is

$$\gamma^2 \, or \left[E/E_0 \right]^2$$

In any storage ring the longitudinal forces, due to the RF systems, are smaller than the transverse forces, which are due to the magnets. In addition γ^2 is large, so we will only consider radiation due to radial acceleration, i.e. Magnetic forces.

The rate of loss of energy = the instantaneous emitted power (P).

where $$P = \frac{2}{3} \frac{rc}{(m_0 c^2)^3} E^2 F^2$$

r = electron radius. m_0 = electron rest mass. F = accelerating force.
c = velocity of light. E = total energy.

But the accelerating force = F = evB = ecB.

$$P = \frac{2}{3} \frac{e^2 r c^3}{(m_0 c^2)^3} E^2 \cdot B^2$$

But remember $(B\rho)=\frac{p}{e}=\frac{E\beta}{ec}$ but $\beta=\frac{v}{c}=1$, where ρ = radius of curvature of the electron's trajectory.

$$\therefore P = \frac{2}{3}\frac{rc}{\left(m_0c^2\right)^3}\frac{E^4}{\rho^2}$$

P = the instantaneous emitted power or the rate of change of energy of the electron itself. In order to find energy loss in one turn we must integrate over 1 turn, i.e. all around the circumference of the accelerator.

But $\int dt = \int \frac{ds}{c}\left[\frac{ds}{dt}=c\right]$, and, as we are only interested in what happens inside bending magnets, we can put $\int \frac{\mathrm{ds}}{\mathrm{c}} = 2\pi\int\frac{\mathrm{dp}}{\mathrm{c}}$.

$$\therefore u = \underbrace{\frac{4\pi}{3}\frac{r}{\left(m_0c^2\right)^3}}_{C}E^4\int\frac{1}{\rho^2}d\rho$$

where u = change in energy over 1 turn.

$$\therefore u = \frac{-CE^4}{\rho}$$

where C is a constant, u = rate of energy loss of the electron, E = electron energy and ρ = electron bending radius inside the dipole magnets

The first consequence if this is that, as we increase the electron's energy, the energy loss per turn increases dramatically. This loss can be offset to some extent by increasing the bending radius of the accelerator, i.e. making the ring larger. However in any electron storage ring we must have the RF system on permanently to compensate for the energy lost per turn due to the emission of synchrotron radiation. Therefore the electrons (or positrons) will be permanently bunched, and will be executing synchrotron oscillations inside the RF bucket. However the emission of the synchrotron radiation will affect these synchrotron oscillations.

Therefore, let us put in some numbers and try to estimate the frequency of the synchrotron oscillations inside the RF bucket.

Any particle inside an RF bucket will execute synchrotron oscillations, see chapter 6. This means that this particle will have a slightly different phase, ϕ, with respect to the RF waveform after each turn around the accelerator, i.e. the phase of each particle with respect to the RF waveform changes in time due to synchrotron oscillations. This change of phase is given by:-

$$\frac{d\phi}{dt} = 2\pi h\Delta f_{rev}$$

where Δf_{rev} = the change in revolution frequency due to the synchrotron oscillations,
h = RF harmonic number, and $d\phi/dt$ = rate of change of phase.

But remember $\frac{df_{rev}}{f_{rev}} = -\frac{dE}{E} \times \eta$ the electrons are ultra-relativistic $\therefore \left(\frac{\Delta E}{E} = \frac{\Delta p}{p} \right)$

$$\therefore \frac{d\phi}{dt} = \frac{-2\pi h \eta}{E} \cdot dE \cdot f_{rev}$$

$$\therefore \frac{d^2\phi}{dt^2} = \frac{-2\pi h \eta}{E} \cdot f_{rev} \cdot \frac{dE}{dt}$$

But dE = change in energy corresponding to the change in revolution frequency Δf_{rev}.
This energy change has two components. The first is due to the energy given to the beam per turn by the RF system. This is given by $V(\sin\phi_T - \sin\phi_s)$, where V = the RF voltage, ϕ_s = phase of synchronous particle, and $\phi_T = \phi_s + \phi$. The second term is just u, the energy lost per turn due to the synchrotron radiation.

$$\therefore dE = \underbrace{V(\sin\phi_T - \sin\phi_s)} - du$$

$$\therefore \frac{dE}{dt} = f_{rev} V(\sin\phi_T - \sin\phi_s) - f_{rev}\, du$$

Combining this with our expression for $\frac{d^2\phi}{dt^2}$ we obtain:-

$$\frac{d^2\phi}{dt^2} = \frac{-2\pi h\eta f_{rev}^2 V(\sin\phi_T - \sin\phi_s)}{E} - \frac{2\pi h\eta f_{rev}^2 du}{E}$$

If we assume *du* is small then $\phi_s \to 0$ and if we assume small energy oscillations then ϕ is small, then:-

$$\frac{d^2\phi}{dt^2} + \left(\frac{2\pi h\eta f_{rev}^2 V \cos\phi_s}{E} \right)\phi + \frac{2\pi h\eta f_{rev}^2 du}{E} = 0$$

This starts to look very like SHM! $\frac{d^2\phi}{dt^2} + \omega^2\phi = 0$,
but we have an extra term.

$$\frac{2\pi h\eta f_{rev}^2 du}{E} = \frac{2\pi h\eta f_{rev}^2}{E} \cdot \frac{du}{dE} \cdot dE$$

We rearrange this "extra term" as follows. Remember $\left(\frac{dE}{E} = \frac{-1}{\eta} \frac{df_{rev}}{f_{rev}} \right)$.

$$\therefore \frac{2\pi h\eta f_{rev}^2}{E} \cdot \frac{du}{dE} \cdot dE = -2\pi h f_{rev} df_{rev} \frac{du}{dE}$$

$\left(\text{But} \frac{d\phi}{dt} = 2\pi h df_{rev} \text{and} f_{rev} = \frac{1}{T_{rev}} \right)$. where T_{rev} = revolution time

$$\therefore \frac{2\pi h\eta f_{rev}^2}{E} \cdot \frac{du}{dE} \cdot dE = -\frac{du}{dE} \times \frac{1}{T_{rev}} \times \frac{d\phi}{dt}$$

Put this into our SHM like equation above and we get:-

$$\frac{d^2\phi}{dt^2} - \left(\frac{du}{dE} \times \frac{1}{T_{rev}} \right) \frac{d\phi}{dt} + \frac{\left(2\pi h|\eta| f_{rev}^2 V \cos\phi_s\right)\phi}{E} = 0$$

This is damped SHM of the form $\left[\phi = e^{-\alpha t} \cos\omega t\right]$

The frequency (ω) = the basic synchrotron oscillation frequency.

$$f_s = \sqrt{\frac{2\pi h|\eta| V \cos\phi_s}{E}} f_{rev}$$

We define ν_S = number of synchrotron oscillations/turn.
= synchrotron tune.

$$\nu_s = f_s / f_{rev} = \sqrt{\frac{2\pi h|\eta| V \cos\phi_s}{E}}$$

The damping time constant (α) is given by:- $\left(\frac{du}{dE} \frac{1}{2 \times T_{rev}} \right)$

where du/dE = rate of change of synchrotron radiation energy loss with energy.

The fact that u (the synchrotron radiation energy loss per turn) varies with energy leads to energy damping, with a damping time constant:-

$$\alpha = \frac{1}{2T} \frac{du}{dE}$$

But:-

$$u = \frac{-CE^4}{\rho} \therefore \frac{du}{dE} = \frac{-4CE^3}{\rho}$$

$$\frac{du}{dE} = \frac{4u}{E}$$

u = energy loss/turn.

$$\therefore \alpha = \frac{2u}{ET}$$

T= revolution time.

E = energy.

The damping time is simply = $1/\alpha$.

NB. This is not 100% correct, since ρ also varies with energy, but it gives a reasonable estimate of the damping time.

$$\text{Damping time} = \frac{1}{\alpha} = \frac{ET}{2u}$$

If we do it correctly we get:-

$$\alpha = \frac{u}{2ET}(2 + D)$$

Where D is property of guide field.

So now we see that the energy spread is damped with a damping time constant which is proportional to $\dfrac{\text{Energy} \times \text{revolution time}}{\text{Energy loss / turn}}$.

But energy loss/turn (u) $\propto$ Energy^4

$\therefore$ The damping time decreases rapidly as we increase energy.

In Figure 1 we show the effect of the damping of the phase of the synchrotron
oscillations. It is obvious that damping of the phase (ϕ) also means damping of the associated energy oscillations, and hence energy spread if the bunch(es).

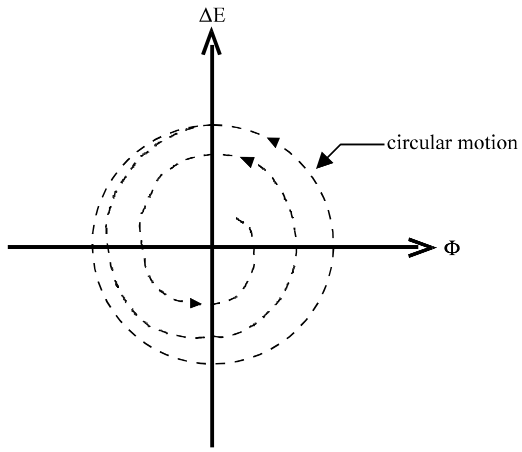
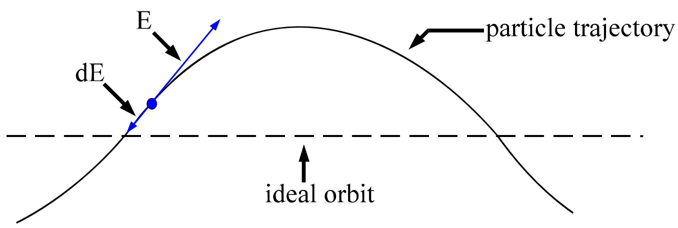


Figure 1: Synchrotron oscillations including the damping

Damping of betatron oscillations.

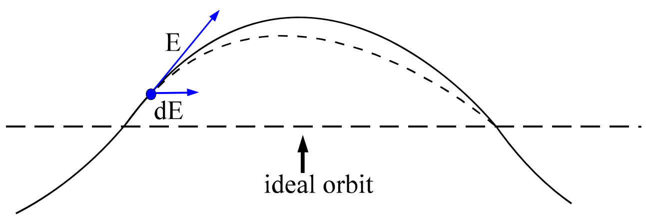
We will start with the vertical case as it is simpler. The emission of synchrotron radiation for a relativistic electron occurs in the direction of motion. See figure 2.



E = electron energy.
dE = loss of energy due to photon emission.

Figure 2: Effect of a vertical betatron oscillation on the emission of synchrotron radiation photons.

On average photon emission will not change the amplitude of betatron oscillation, since half of the photons will be emitted with a small upward angle and half will be emitted with a small downward angle. However, the RF system always supplies energy exactly parallel to ideal orbit. See figure 3.



E = electron energy.
dE = energy gain from RF.

Figure 3: Energy supplied by the RF system with respect to a vertical betatron oscillation

The angle a particle makes with respect to the ideal orbit is given by:

$$\left(y'\right) = \frac{E_{\perp}}{E_{\parallel}}.\quad \text{See figure 4.}$$

$E_{\parallel}$ = total longitudinal energy (along ideal orbit).

$E_{\perp}$ = energy perpendicular to ideal orbit.

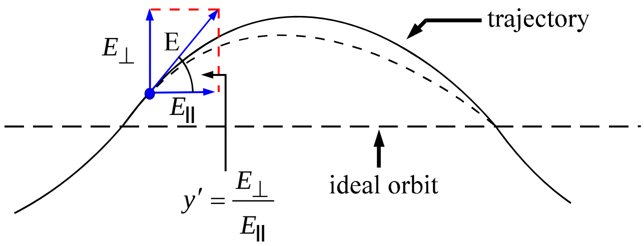


Figure 4: Energy due to a transverse betatron oscillation

Now the RF system increases $E_{\parallel}$ without changing $E_{\perp}$.

$$\therefore \text{ new } y' = \frac{E_{\perp}}{E_{\parallel} + dE_{\parallel}} = \frac{E_{\perp}}{E_{\parallel}}\left(1 - \frac{dE_{\parallel}}{E_{\parallel}}\right) \qquad \left(dE_{\parallel} \text{ is small}\right).$$

$$\therefore \text{ new } y' = y'\left(1 - \frac{dE}{E}\right).$$

$$\therefore \text{ Change in angle} = -y'\frac{dE}{E} = \mathrm{dy}'\ .$$

Therefore the energy gain from RF reduces the angle of betatron oscillation.

But a change in *y'* alters the amplitude of the oscillation. To see this we can look at the "normalised" phase space diagram. See figure 5.

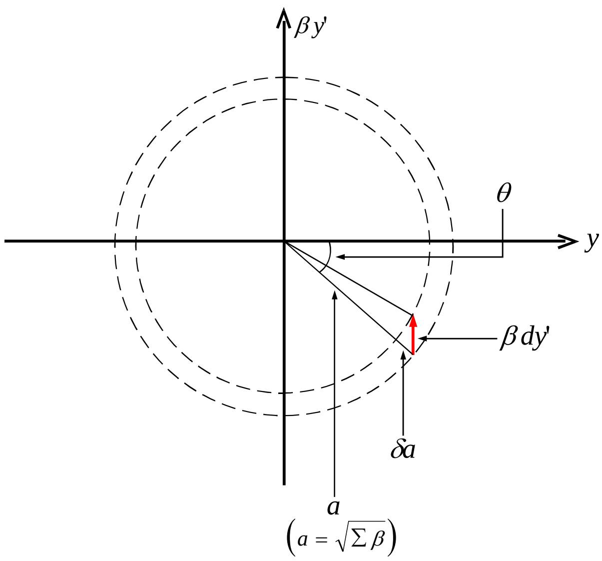


Figure 5: Vertical normalised phase space diagram showing the effect of a small change in y'

$$da = \beta\, dy' \sin\theta$$

$$da = -\beta\, y' \frac{dE}{E} \sin\theta$$

But we must sum over all possible values of θ from 0→2π.
Over many photon emissions, the overall change of amplitude <da> is given by:-

$$\langle da \rangle = -\sum_{\theta=0}^{2\pi} \beta\, y' \frac{dE}{E} \sin\theta$$

But $\beta y' = a \sin\theta$

$$\therefore \langle da \rangle = -a \frac{dE}{E} \sum \sin^2\theta$$

$$\therefore \frac{\langle da \rangle}{a} = -\frac{1}{2}\frac{dE}{E}$$

$$\therefore \frac{\langle da \rangle}{a} = -\frac{1}{2}\frac{\Delta E}{E}$$

But ΔE is the change in energy over 1 turn, remember we summed θ from 0→2π or many photon emissions at all possible values of θ from 0→2π.

$$\therefore \Delta E = u$$

$$\therefore \frac{\langle da \rangle}{a} = \frac{-u}{2E}$$

$$\text{(Over 1 turn)} \rightarrow \underbrace{\Delta a = \frac{-u}{2E} a}_{\Downarrow} \leftarrow \text{(this is an exponential damping } \Delta a\ \alpha\ a\text{)}.$$

$$\therefore \frac{da}{dt} = \frac{-ua}{2ET} \rightarrow \left(a = a_0 e^{-\alpha t}\right) \quad (T = \text{revolution time})$$

$\therefore$ Damping coefficient is $\alpha = \dfrac{u}{2ET}$.

This result is very similar to longitudinal case, which is reassuring! In fact the vertical damping coefficient is approximately half of the longitudinal one.

<u>**Horizontal betatron oscillations**</u>

Vertically we saw that there is a change in betatron amplitude, which is given by $\frac{\Delta a}{a} = -\frac{u}{2E} = -\frac{dE}{2E}$.

However, in the horizontal case, we must also consider the change in radial position due to the change in energy $\left(\frac{dr}{r} = \eta\frac{dE}{E}\right)$. This change in energy means that as the particle loses energy it's ideal orbit changes, and this increases it's betatron oscillation amplitude. See figure 6.

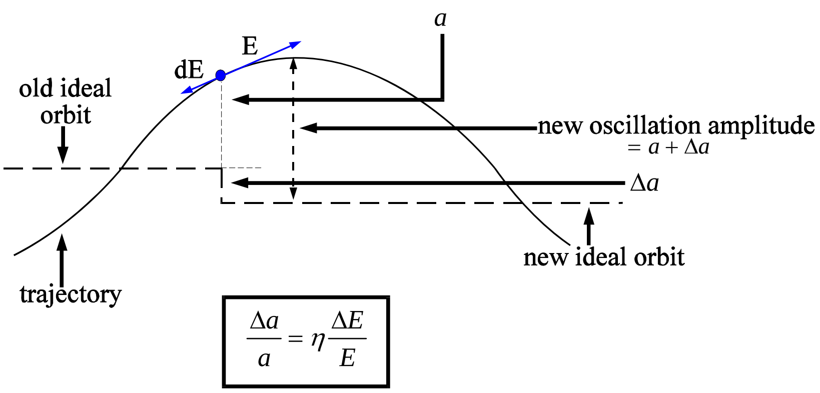


Figure 6: Change in horizontal trajectory due to emission of a photon.

So horizontally we must include this increase in the oscillation amplitude. This means that the change in betatron oscillation amplitude is given by:-

$$\therefore \frac{\Delta a}{a} = +(1-2\eta)\frac{dE}{2E}$$

Now we can proceed exactly as before, and we get a damping coefficient.

$$\therefore \alpha_x = +(1-2\eta)\frac{u}{2E}$$

$\therefore$ Horizontal damping is slower.

$\left(\text{In fact this is rather oversimplified, the real result is } \alpha_x = +(1-\eta)\frac{u}{2ET}\right)$.

However, provided η is small, the damping is still present, but there is a "heating" term $\frac{\eta u}{2ET}$, which will limit the horizontal emittance.

This damping of betatron oscillations and energy spread is very important.

$\frac{\Delta E}{E}, \varepsilon_H, \varepsilon_V$ all shrink in time, with typical damping times between a few milliseconds and a few seconds. This means that the beam gets smaller, which is general very useful! This will reduces losses. Any oscillations due to injection will get rapidly damped away, which will make injection into the accelerator easier, and any oscillations due to resonant type instabilities will be damped provided that the instability growth time is greater than the damping time. This makes electron machines very forgiving, but they do need lots of RF power, which will lead to other bunched beam instability problems at high intensities.

Quantum fluctuations

What stops the energy spread (dE) and vertical beam size damping to zero?

The emission of photons is a random process!! Whenever a photon is emitted the energy of the electron changes slightly, and this induces a small synchrotron oscillation. If we add many such, small, random changes like this, the amplitude of the synchrotron oscillation will grow. This is a statistical process, which is the same as a random walk.

Imagine a man (or woman), who may have had a little too much to drink, leaning against a lamp post. If he (or she) takes random steps in any direction, as inebriated people are supposed to do. He (or she) will, after many such steps, finish a certain distance from the lamp post. For a large number of steps this distance will be proportional to the square root of the number of steps taken. Therefore the emission of many photons will cause the energy of any single particle to deviate from the mean beam energy, which will lead to an increase in overall energy spread. This growth is limited by the longitudinal damping, but it does mean that when the "growth rate = damping rate", the energy spread of beam will stay constant. However the energy of each lepton will oscillate back and forth around a certain mean energy, as it emits photons and receives energy from the RF system. Therefore the stable electron bunch will have a finite length and energy spread.

A typical fluctuation in the particle energy comes from the finite fluctuation in the number photons emitted in one damping time.

If E_P = energy of 1 photon.

Remember that the longitudinal damping time = ET/2u seconds.

or the longitudinal damping time = E/2u revolutions.

The energy loss per turn = u.

$\therefore$ The number of emitted photons per turn = u/E_p .

$\therefore$ Number of photons emitted in 1 damping time is given by:-

$$(u/E_p)*(E/2u) = (E/2E_p)$$

The energy fluctuations in the lepton energy are given by the rms fluctuation in the number of photons emitted per turn i.e.

$$\sqrt{(E/2E_p)}$$

But each photon has an energy E_p and therefore the rms energy deviation $\propto \sqrt{EE_P}$.

Now it happens that the average emitted photon energy $E_p \propto$ Electron Energy^3:-

$$E_p \propto E^3$$

$\therefore$ The energy spread induced in the bunch by the quantum fluctuations is proportional to the lepton energy squared.

Therefore the relative induced energy spread, σ_E, is given by:-

$$\sigma_E/E \propto E$$

As a result of the quantum fluctuations there will always be a minimum energy spread σ_E that can be obtained, which will be proportional to the beam energy squared.

In the transverse planes, we already have seen that we have a horizontal heating term due to the non-zero dispersion function. Vertically there is, however, no equivalent term, and the vertical emittance can, and does get very small. It is however limited by coupling of horizontal and vertical betatron motion.

NB: In all our discussion the damping is due to the gain of energy from the RF system, not the emission of synchrotron radiation itself.

But in all cases the damping time $\propto \frac{ET}{u}$.

∴ If we increase u (the energy loss/turn) by adding extra dipole magnets, which will increase synchrotron radiation emission, we can decrease the damping time, which leads to a significant reduction in emittance and hence beam size. These extra magnets are called "Wigglers", and they consist of a chain of small dipoles, which deflect the beam transversely backwards and forwards. This increases the number of photons emitted per turn, without changing the overall particle trajectory.

Vertically, this is fine and the vertical emittance is reduced. Horizontally, it is also true, but one must be careful as a change in energy, due to emission of a photon, leads to a change in radial position, which in turn will increase in the horizontal oscillation amplitude.

$$\left(\text{Remember: dx} = \underbrace{D(s)}_{\text{(dispersion fonction)}} \frac{dE}{E} \right)$$

Therefore in the horizontal plane one must place the Wiggler where *D(s)*=0.0 then it will not excite horizontal oscillations, and will decrease rather than increase the horizontal emittance.
Longitudinally, the Wiggler increases number of photons emitted, and therefore it will increase the quantum fluctuations. So the introduction of a Wiggler will increase the energy spread, whilst decreasing transverse emittances.

Chapter 8: Transfer lines, injection and ejection.

We treat particle trajectories and motion in a transfer line in exactly the same way as we treated particle trajectories in our accelerator/storage ring. There are dipole magnets for deflecting the beam, and quadrupoles for both horizontal and vertical focusing.
The transverse particle motion is again described in terms of transverse phase space using betatron oscillations, and β-functions etc. and the beam energy spread again leads to dispersion functions.
The same 2×2 matrices are used to describe individual particle motion.

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}.$$

But there is one very important difference between a transfer line and a "circular" accelerator/storage ring. The transfer line is **not** closed. This closure was very important in the accelerator as it allowed us to calculate the Twiss parameters, by equating the initial and final Twiss parameters for a complete turn around the machine. For a transfer line the initial lattice parameters ≠ final lattice parameters.

∴We get a rather more complicated transfer matrix (M).

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu + \alpha_1 \sin\mu\right) & \sqrt{\beta_1\beta_2}\sin\mu \\ \frac{\left(1+\alpha_1\alpha_2\right)\sin\mu + \left(\alpha_2 - \alpha_1\right)\cos\mu}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu - \alpha_2 \sin\mu\right) \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}.$$

NB: If we put $\beta_1 = \beta_2 \quad \alpha_1 = \alpha_2$ etc. then this reduces to the matrix we had for our storage ring, but for a transfer line we must retain the full matrix.

As before we calculate β etc. by equating this matrix with the result of multiplying all the individual matrixes for each successive element in the transfer line. There will an infinite number of solutions for our transfer line, since for any value β_1 there will a particular solution for β_2. I.e. the final β_2 depends on the choice of β_1. This is the case for all the Twiss parameters,
By definition, the transfer line transports the beam from one point to another, from one accelerator to another or from an accelerator into an experimental set-up the. The initial Twiss parameters (β_1 etc.) will be determined by the accelerator, from which the beam is being extracted.

The final Twiss parameters (β_2 etc.) will determined by the needs of the accelerator or experiment receiving the beam.

Imagine that we are trying to calculate a transfer line in order to transfer the beam from machine (1) to machine (2). See figure 1. The beam in machine (1) is described by a phase space ellipse, which rotates as the beam moves around the machine. At the point A, at which the beam is extracted from machine (1), this ellipse will have some particular orientation. See figure 2.

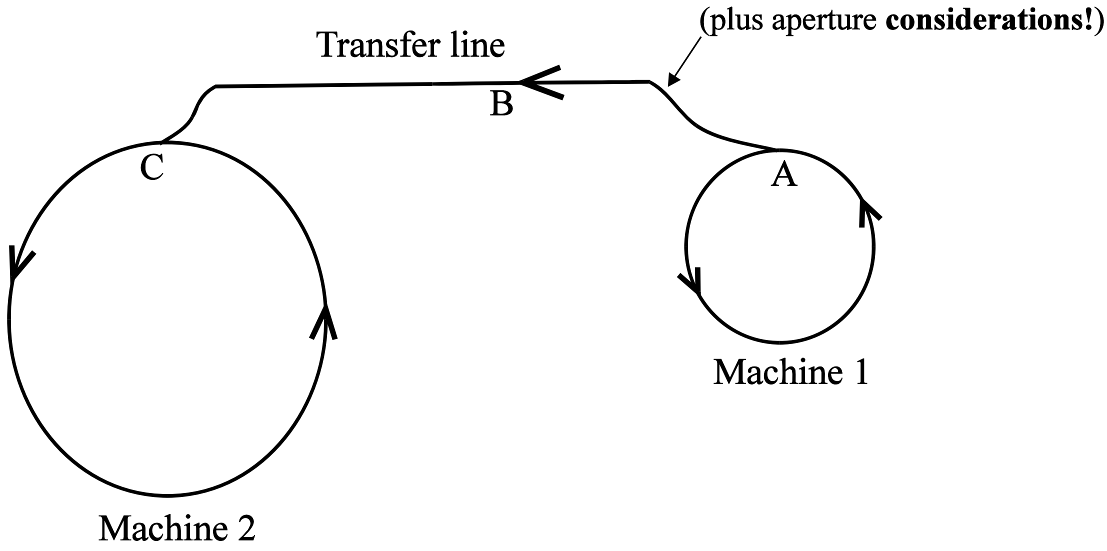


Figure 1: Layout for our transfer line

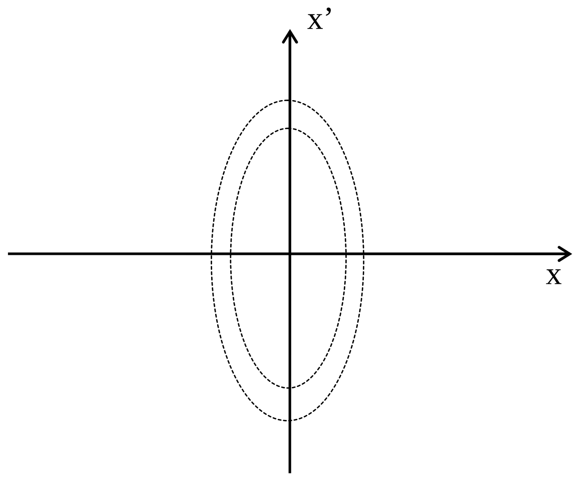


Figure 2: Transverse phase space ellipse at extraction (A) from machine (1)

Now we can use the transport matrix, which describes our transfer line to transport this ellipse to machine (2) at point C. But machine (2) will have it's own predetermined transverse phase space ellipse at C, and if the phase space ellipse, which arrives from the transfer line B, is different (which is very likely) then what will happen to the beam?

This situation is shown in figure 3. Once the particles arrive in machine (2), they will follow the phase space ellipse as dictated by the lattice of machine (2). If the phase space ellipse, arriving from the transfer line does not “match” that of the machine (2) at point C, then this will lead to an increase in the area of the ellipse, or an increase in transverse beam emittance. This increase in transverse oscillation amplitude could also easily lead to a loss of particles over first few turns.

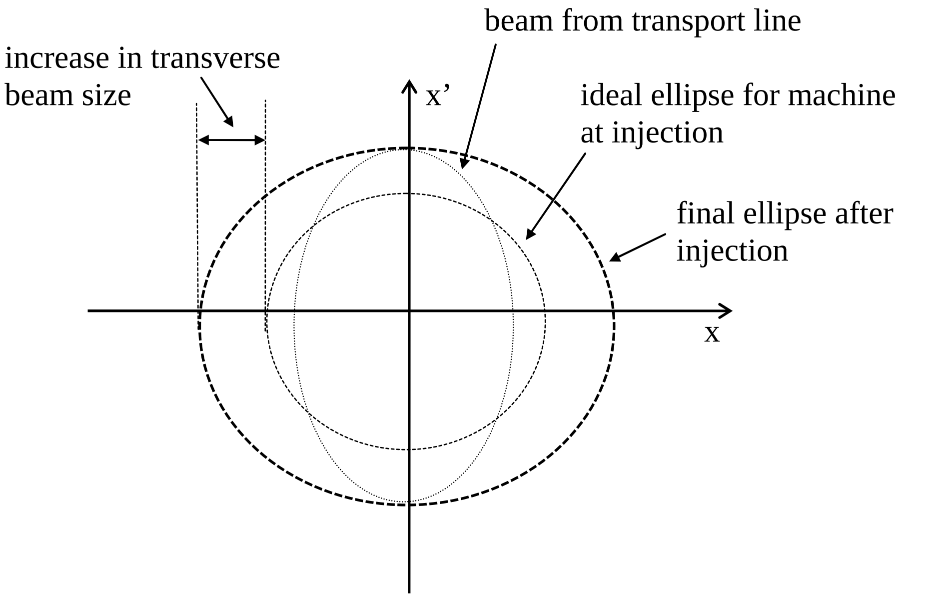


Figure 3: Emittance blow-up as a result of a mismatched transfer

Therefore when we design the transfer line to take beam from machine (1) $\rightarrow$ (2), we must set $\beta_1\ \alpha_1 ...$ for the transfer line equal to the $\beta\,\alpha ...$ values in machine (1) at point A. Then the transfer line is calculated so that at point C we obtain $\beta_2\ \alpha_2 ...$ for the transfer line to be equal to the $\beta\,\alpha ...$ values for machine (2) at point C. This process is called **matching**. In the calculation process we must also take care that the beam envelope is small enough for the expected beam emittances in the transfer line to avoid losing particle inside the transfer line. This requires a detailed calculation of $\beta\,\alpha ...$ at all critical limits. Due to the constraints of the matching conditions outlined above, the quadrupole excitation is usually rather complex to calculate in a transfer line. The variables in the calculation are the positions and strengths of quadrupoles..

Correction of trajectory.

Exactly as for our accelerator there will be trajectory errors, which will need correction. These errors will be due to quadrupole misalignments,

energy changes, changes deflection angles in dipoles, and errors in angle and/or position at injection into transfer line.
Transfer lines are normally equipped with various forms of beam position monitors, such as position sensitive pick-ups, luminescent screens etc. which give transverse position and beam envelope information. There are also small horizontal and vertical dipoles to correct trajectories.

Injection / Ejection.

Obviously in order to transfer a beam into or out of an accelerator we need to deflect the beam into or out of the storage ring, with a minimal particle loss.

For injection we want to inject the beam as closely as possible on to the "closed orbit" of the machine. Otherwise the injected bunch will perform large amplitude betatron oscillations around the closed orbit. This is very undesirable as it leads to an increase in the beam emittance, and, in extreme cases, can easily lead to large losses over a very few turns. For injection we need two elements, which deflect the beam transversely, (Septum + kicker magnets). The injection process can take place in either the horizontal or the vertical plane. See figure 4.

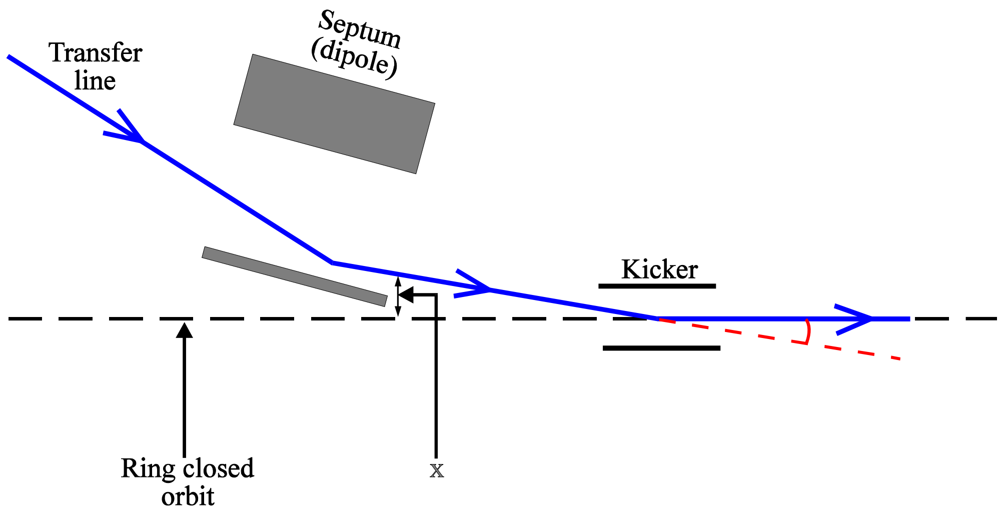


Figure 4: Schematic layout of a single turn injection

The septum magnet is simply a closed dipole magnet, which only deflects the beam in the injection channel and does not affect the beam circulating in the storage ring. This septum then deflects the beam from the injection channel into the kicker magnet, which is in the storage ring. Now the kicker deflects the beam, in order to place it exactly onto the closed orbit of the machine. Therefore the angle needed for the kicker is, see figure 4:

$$\delta = \frac{x}{\sqrt{\beta_s \beta_k} \sin \mu_x}$$

where $\beta_s \beta_k = \beta$ functions at septum and kicker.
$\mu_x =$ phase advance septum $\rightarrow$ kicker.

In order to keep δ small (i.e. reduce the power need for the kicker) we try to get $\beta_s\ \beta_k$ large and $\sin \mu_x$ large. This later condition is very important as it says that the phase advance (μ_x) must be very close to $\pi/2$, or $3\pi/2$, or $5\pi/2$ etc.

$$\therefore \mu_x = (2n+1)\frac{\pi}{2}$$

There is one further complication, we must synchronise the kicker pulse with the injected bunch, since the beam will pass through the kicker on every turn around the storage ring and therefore the kicker must be a fast pulsed element, which only deflects the 1^{st} turn and not 2^{nd}. See figure 5. This is not true for the septum magnet, through which the beam only passes once on its way into the ring. Although the septum may be a pulsed element in order to save energy as it is only used at injection.

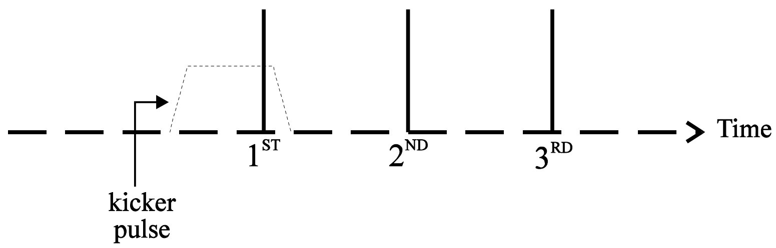


Figure 5: Synchronisation of the kicker pulse with the incoming beam bunch

If it is necessary to maintain the beam in a bunched form for a certain time, when it arrives in the storage ring. This requires that the RF system in the ring receiving the beam should be on, and synchronised with the bunch, when the bunch arrives, in order to transfer the bunch directly into a predetermined RF bucket. Since it is not possible to synchronise the RF system with the bunch before the bunch actually arrives! It is usual to synchronise the RF system of the receiving machine to that of the sending machine, just before the beam transfer.

<u>Correction of injection.</u> Or how do we optimise the septum and kicker settings to ensure a good, efficient injection?

For this it is necessary to measure the injection error. There are several methods the simplest uses 2 beam position monitor (BPM) signals in the ring. See figure 6.

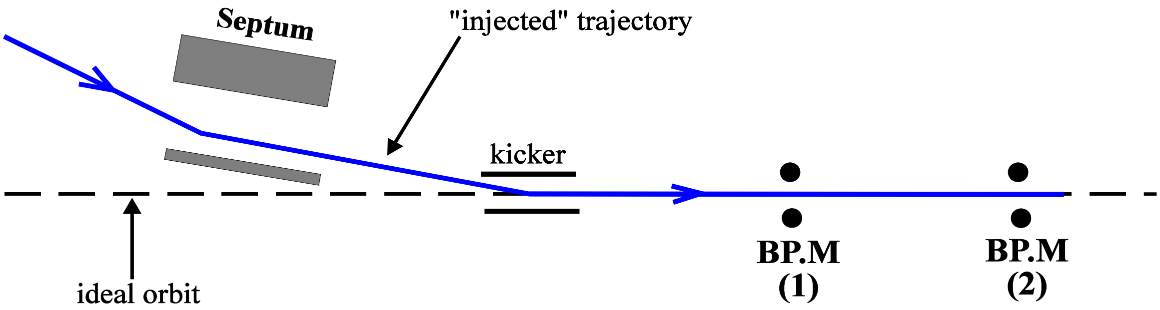


Figure 6: Measurement of injection errors using two BPMs

Remember that an injection error will result in a transverse oscillation around ideal orbit. The phase advance from the injection septum to the kicker ≈ 90°.
Therefore, in figure 6, we choose BPM(1) so that the phase advance (ϕ) from the kicker to BPM1 = 90°. Likewise we choose BPM(2) so that ϕ from the septum to BPM(2) = 270°.
Now an error on the injection kicker setting translates directly into a position error at BPM(1), and an error on the septum setting translates into a position error at BPM(2). See figure 7.

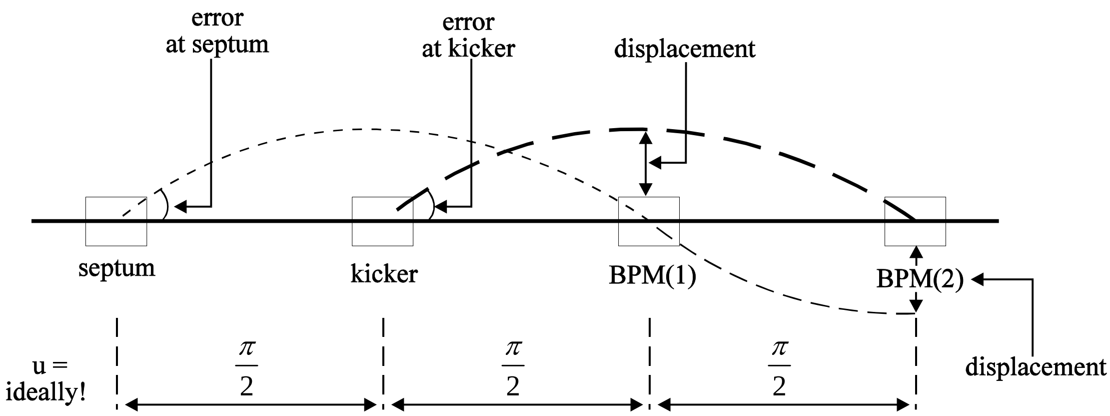


Figure 7: Optimum phase advances for injection error measurement

To calculate the actual corrections needed we use the same formula that we found for calculating the amplitude of orbit deflections in the storage ring:

$$x = \delta\sqrt{\beta_1\beta_2}\sin(\mu_x)$$

x = displacement correction.

δ = angle at correction element.

For a horizontal injection the vertical error is usually less important, however small dipoles at the end of the transfer line are used to ensure that the injected beam arrives correctly in the vertical plane in the storage ring.

There is another method of measuring the injection error. If we observe the transverse position signal from a single BPM on a turn by turn basis as the injected bunch makes its first turns in the ring. We see that the beam oscillates transversely around some mean position. See figure 8.

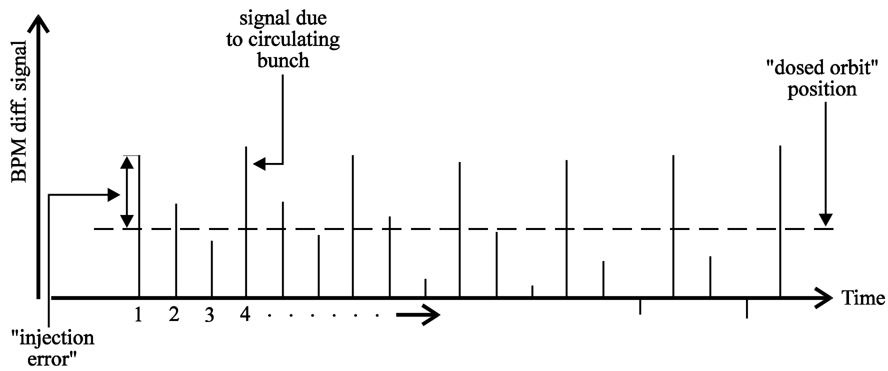


Figure 8: Transverse position over the first 20 turns after injection

This mean position is just the closed orbit displacement at the BPM, and the oscillation is due to an incorrect injection into machine. From the observed difference, and knowing the phase advance of the BPM from the septum and the injection kicker, it is possible to calculate the injection correction needed as before. An ideal injection would show no oscillation. See figure 9:

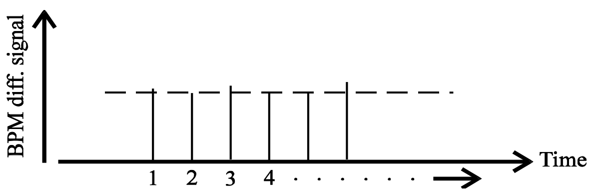


Figure 9: BPM signal for an optimised injection

Single turn ejection is exactly the reverse of single turn injection, again using a fast kicker and septum magnet, but with the beam direction reversed. So the kicker deflects the beam off of the closed orbit into the gap of the septum magnet, and the septum deflects the bunch into the extraction line. Again care must be taken to exactly synchronise the ejection kicker, with the circulating bunch to be ejected.

Multi-turn injection

We have only considered single turn injection until now. However, to increase the injected current, it is possible to inject more than a single turn into the ring. This is called multi-turn injection. This process is shown in figure 10.

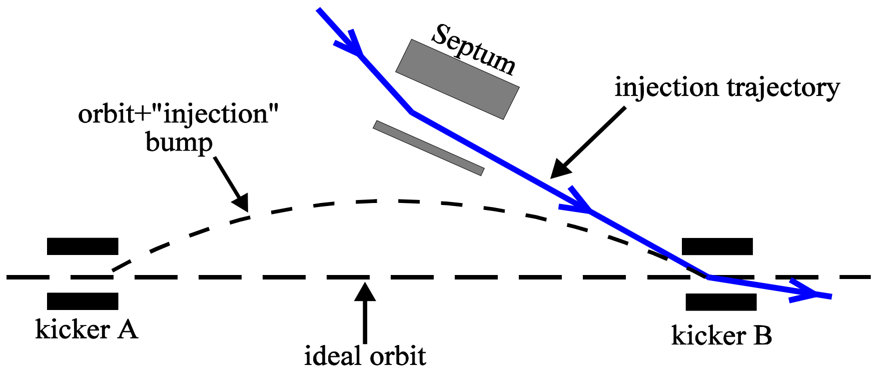


Figure 10: Layout for a multi-turn injection scheme

First we will consider a lepton ring, with a bunch already circulating inside the machine.
We use two kickers to create an orbit bump inside the machine. This bump moves the circulating beam very close to the injection septum. Now the injected bunch arrives and is deflected by the septum to kicker B. This bunch is then injected but with a small angle error. Now if the kickers are turned off the beam and the injected bunch move back to the ideal central orbit with the injected bunch oscillating around the stored beam, but, for leptons, these oscillations are soon damped, as we saw in chapter 7. Therefore, after a few damping times, it is possible to repeat the procedure.
The phase advance from kicker 1 to kicker 2 must be exactly π or the “fast bump” will affect rest of the machine. For this reason 3 kickers are often used so that the injection process can be adjusted for several different machine “optics”.

One can repeat the same procedure for protons, but there is no damping. In this case the fast bump is turned down slowly over a few turns (<20). Beam is then injected with ever increasing betatron oscillation amplitude until all the available transverse phase space is filled. For protons this increases the injected beam intensity, but at the expense of increasing the transverse beam emittance.

Multi-turn ejection

Often when ejecting beam into an experimental detector, some form of "spill" is needed as the data rate that the experiment can handle is limited. If this is the case some form of slow multi-turn ejection is required. This is done for the fast majority of high energy physics accelerators, e.g. 100's milliseconds for the PS, a few seconds for the SPS and, a rather special case, several thousand seconds for LEAR. In each instance the extraction method is the same.
The slow extraction uses a standard septum magnet system, (in fact there are often two, one electro-static and one magnetic, as the electro-static septum, which deflects the beam into the larger magnetic septum, can be made very thin to reduce particle losses). However, instead of a transverse kicker to deflect the circulating particles into the electro-static septum gap, the particles are moved onto a horizontal betatron resonance, (normally third order). Once the particles are trapped in the resonant condition, their horizontal oscillation amplitude rapidly grows, causing them to jump into the septum gap and be extracted from the machine. Figure 11 shows what happens in transverse phase space to a particle, during the resonant extraction process.

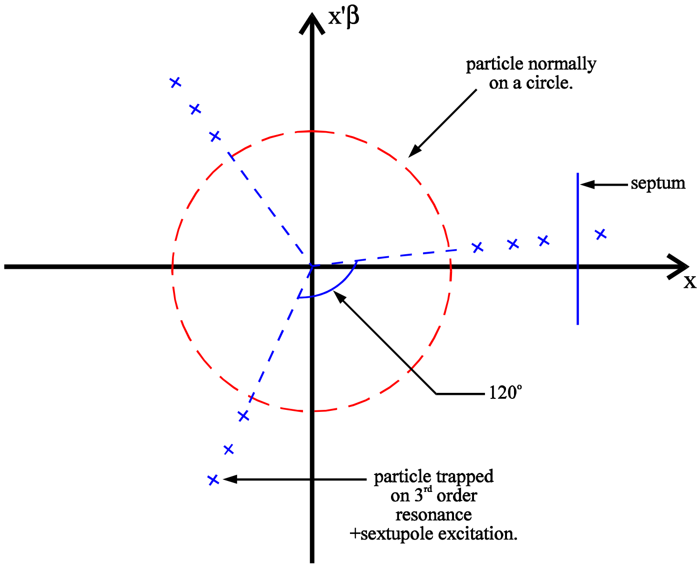


Figure 11: Phase space plot for a particle during resonant extraction

We have seen in chapter 5 that sextupoles excite 3rd order resonances. Therefore we use sextupoles to drive the 3rd order extraction resonance. To ensure that the particles reach the electrostatic septum before they get lost on the vacuum chamber, a local orbit bump is applied to ensure that the septum is the horizontal aperture limitation in the machine during the extraction process.

So the slow extraction is done in the following fashion. The non-integer part of Q_h (q_h) is set very close to very close to 0.333, then the sextupole excitation of this resonance is put on and a horizontal orbit bump at the extraction septum is applied. Now the rate of extraction can be controlled by moving the resonance, $q_h = 0.333$, into the beam or moving the beam onto the resonance. The extracted beam intensity is determined by the initial beam intensity and the length of the extraction process, which is defined by the rate at which the particles are moved onto the extraction resonance. For the first option, the machine quadrupoles are changed to change the tune of the machine and sweep the extraction resonance into the circulating beam. This gives controlled spills of up to a few seconds in length. The second option is slightly more complicated, as it requires a change in the chromaticity of the machine. Remember that the sextupoles control the chromaticity, or the relationship between particle momentum and betatron tune:

$$\frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$

By adjusting the sextupole settings, it is possible to set $\xi_h \neq 0.0$. If the chromaticity is set to be slightly positive accelerating the particle will increase there betatron tune. Now accelerating the beam will move particles onto the resonance, and these particles will be extracted. This method is used for longer spills, and, at LEAR, this type of extraction has given constant intensity spills of up to 3 hours 20 minutes in length.

Chapter 9: Longitudinal Beam Instabilities

Until now we have concentrated upon incoherent motion, in which each particle moves independently of all the others. Now we will look at coherent effects, i.e. what happens when all the particles move in phase in response to some external excitation. These effects are very easy to observe, and very destructive. All particle accelerators will have some problems with them at some time.
Why do coherent instabilities arise? The circulating beam, will induce some Electro-Magnetic fields in the vacuum chamber, which will act back upon the particles in the beam. Now a small perturbation to the beam will induce a change in these fields. If this change tends to amplify the perturbation, this will lead rapidly to instability.

Induced fields

Figure 1 shows schematically how a circulating bunch induces an image current in the vacuum chamber surrounding the beam.

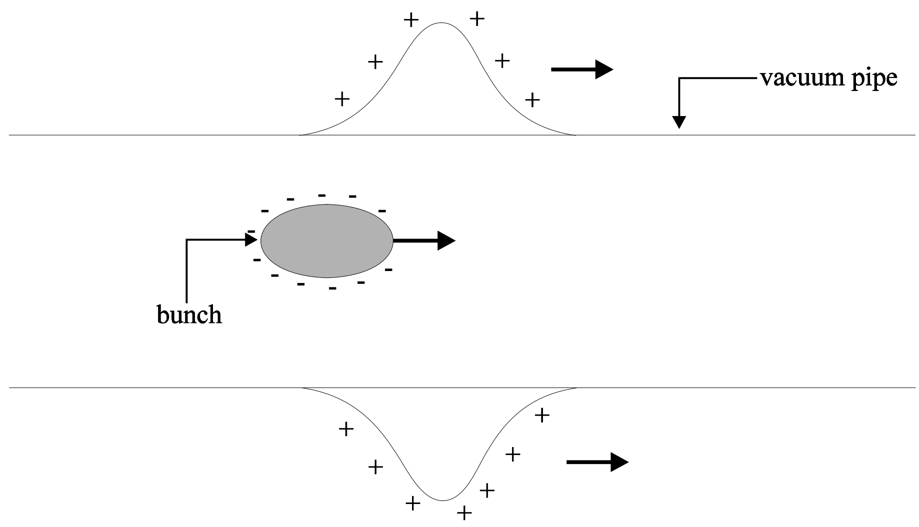


Figure 1: Vacuum chamber image currents due to a circulating bunch

This will only be a very superficial look to explain the terminology and the basic ideas. We will avoid Maxwell, Lorenz... etc.
The beam induces charges on inner surface of vacuum chamber, which circulate with the bunch and form an image current, with same magnitude but opposite sign to beam current. However, due to changes in vacuum chamber material, shape, cross section etc. the chamber presents an impedance to this induced current. This impedance can be resistive, capacitive or inductive, and is expressed as Z.

$$\therefore Z = Z_r + iZ_i$$

The image current combined with this impedance induces a voltage, which affects the charged particles in the bunch. The magnitude of this effect depends on the bunch intensity (current) and the vacuum chamber itself (impedance).

$\therefore$ All instabilities are intensity dependant.

Now we will have to analyse the fields induced by a small deviation from the nominal bunch position and/or particle distribution. Then see if the resulting induced voltage amplifies the deviation (instability), or, if the induced voltage reduces the initial deviation (stability).

Longitudinal Instabilities

Any change in cross-section of vacuum chamber leads to a finite impedance. So for longitudinal motion it is usual to describe the vacuum chamber as a series of cavities, for each change in material, shape or size. The beam induces a voltage in each such a cavity, which will decelerate the particles. If the cavity has some resonant frequency which corresponds to some harmonic (multiple) of the revolution frequency then this will lead to a large induced Electro-Magnetic field, which in turn will mean that this particular cavity will have a large effect on the beam.
The longitudinal instability mechanism is outlined in figure 2.

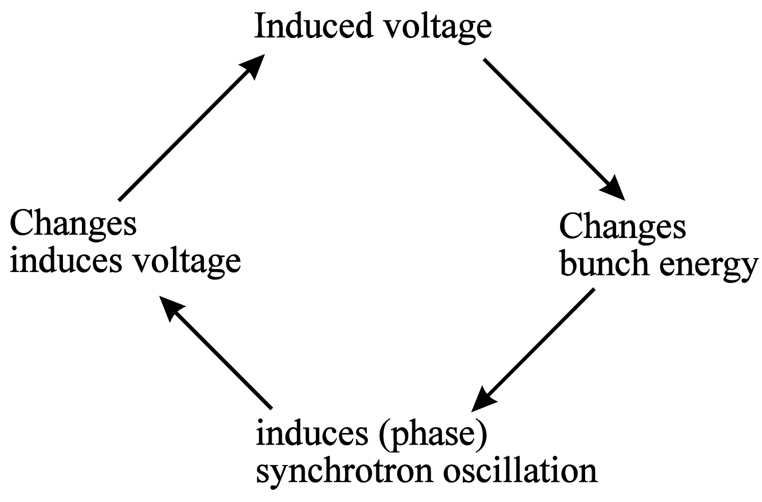


Figure 2: Longitudinal instability mechanism

For any "cavity", two frequencies are important.
ω = driving or bunch revolution frequency
ω_R = resonant frequency of "cavity".

If $n\omega = \omega_R$ then the cavity has a resistive impedance (n = integer) and the induced voltage will be in phase with the excitation due to the circulating bunch.

If $n\omega < \omega_R$ then the cavity has an inductive impedance, and the induced voltage lags behind the excitation due to the circulating bunch

If $n\omega > \omega_R$ then the cavity has a capacitive impedance, and the induced voltage leads the excitation due to the circulating bunch

This is shown in figure 3.

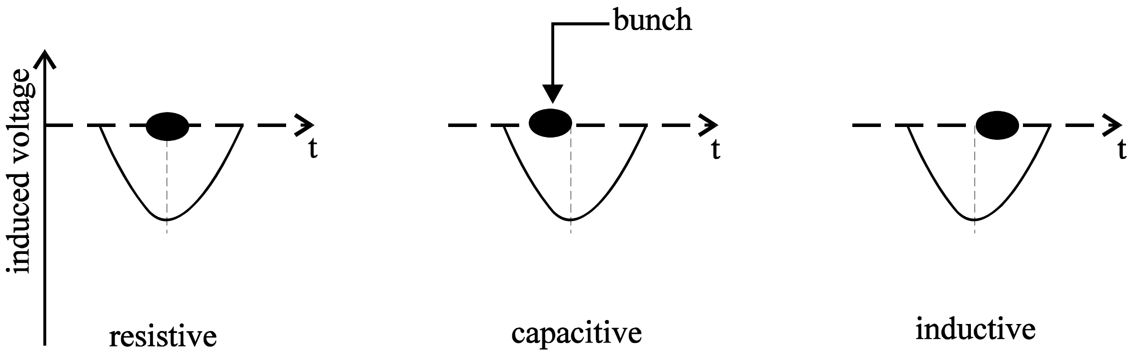


Figure 3: Induced voltage as function of impedance type

We divide longitudinal impedance into two classes.
1) Narrow band high-Q resonators. — RF cavities tuned to some harmonic of the revolution frequency.
2) Broad band cavities — basically rest of the machine.

The RF cavities are most important source of longitudinal impedance in machine, but we must not forget the rest of the machine.

Let us start by looking at a single bunch, with revolution frequency = ω. This bunch is passing through an RF cavity, which resonates at frequency ω_R ($h\omega \approx \omega_R$).
Firstly let $h\omega < \omega_R$, see figure 4, and now let us induce a coherent synchrotron (phase) oscillation.
As the bunch oscillates around the synchronous position it will alternately lose and gain energy, which will increase and decrease it's revolution frequency (ω). But the impedance seen by the beam determines the voltage the bunch induces in the cavity, and this impedance is a function of frequency, see figure 4. We are only concerned with voltage induced in the cavity, by the bunch itself, not with the voltage that may be applied externally to the cavity.

However, in figure 4, an increase in the cavity impedance as seen by the beam leads to an increase in the induced voltage, which increases the energy lost by the beam in the cavity.

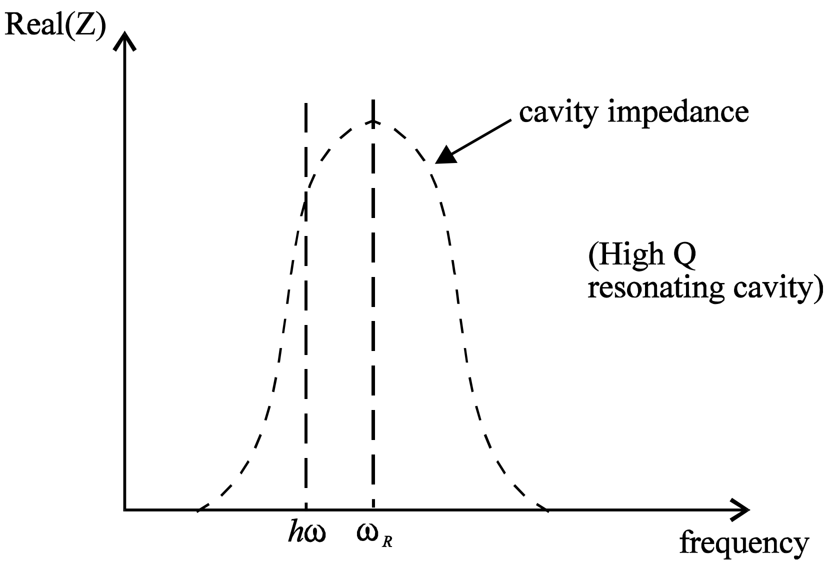


Figure 4: Impedance as a function of frequency. Note the revolution frequency is just below the resonant frequency of the cavity.

If we consider in detail the case shown in figure 4 then, as the beam energy increases the revolution frequency decreases and the energy lost in the cavity creases. If the beam energy decreases then the revolution frequency increases and the energy loss in the cavity increases. Therefore under these conditions the cavity tends to increase the energy oscillations of the particles, which is unstable.
I.e. the action of the induced voltage increases the energy deviations.

If we re-tune the resonant frequency of the cavity so that $h\omega > \omega_R$, see figure 5, then as the bunch energy increases the energy loss in the cavity increases, and as the bunch energy decreases the energy loss in the cavity decreases. This tends to damp the instability. This is known as the Robinson Instability, and to avoid it we must tune the resonant frequency of all the RF cavities correctly.

The Robinson Instability is a single bunch dipole mode oscillation. I.e. The whole bunch is moving back and forth around the synchronous position as a single solid object. Figure 6 shows how this instability would appear both in longitudinal phase space and in time. The later is very useful as it is what one would see if one observed this instability directly on an oscilloscope. This mode of oscillation is the simplest and is called a "mode 1" oscillation, and obviously it's frequency is just the synchrotron frequency.

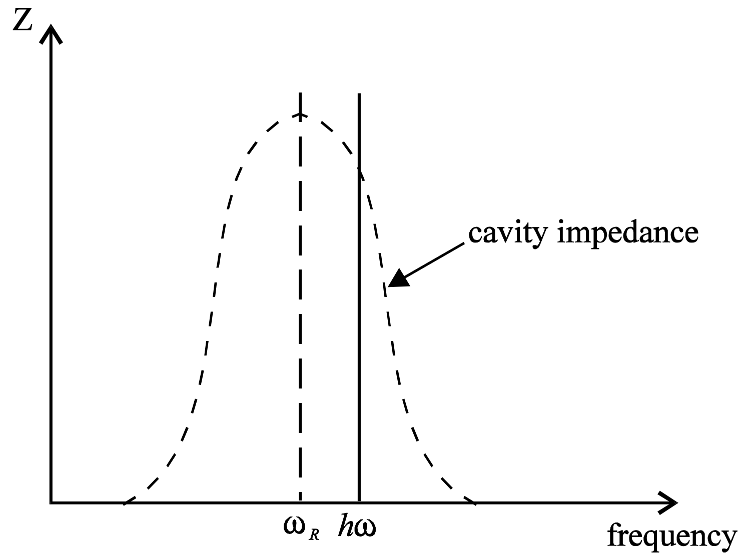


Figure 5: Impedance as a function of frequency. Note the revolution frequency is now slightly higher than the cavity resonant frequency.

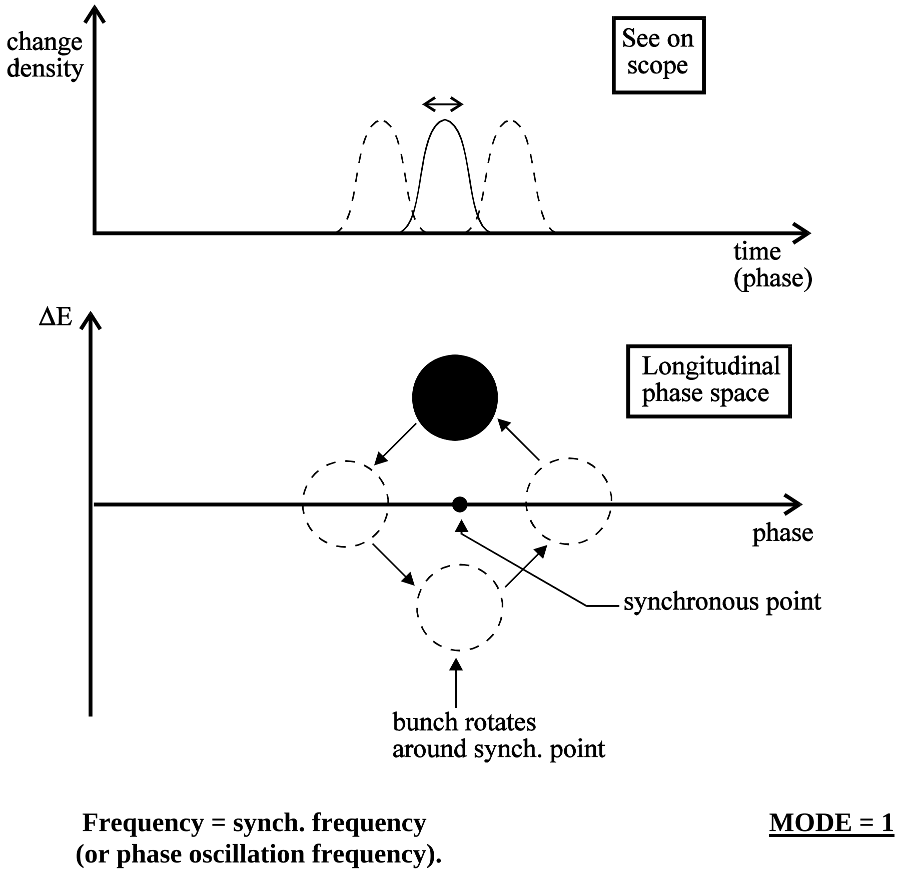


Figure 6: Single bunch dipole instability

It is possible to excite higher order modes, which will also be driven by resonant cavities in the accelerator, e.g. Quadrupole mode, see figure 7. In this mode the centre of gravity of the bunch does not move but the bunch lengthens and shortens. In figure 7 we can see that the frequency of this mode 2 oscillation is twice the synchrotron frequency.

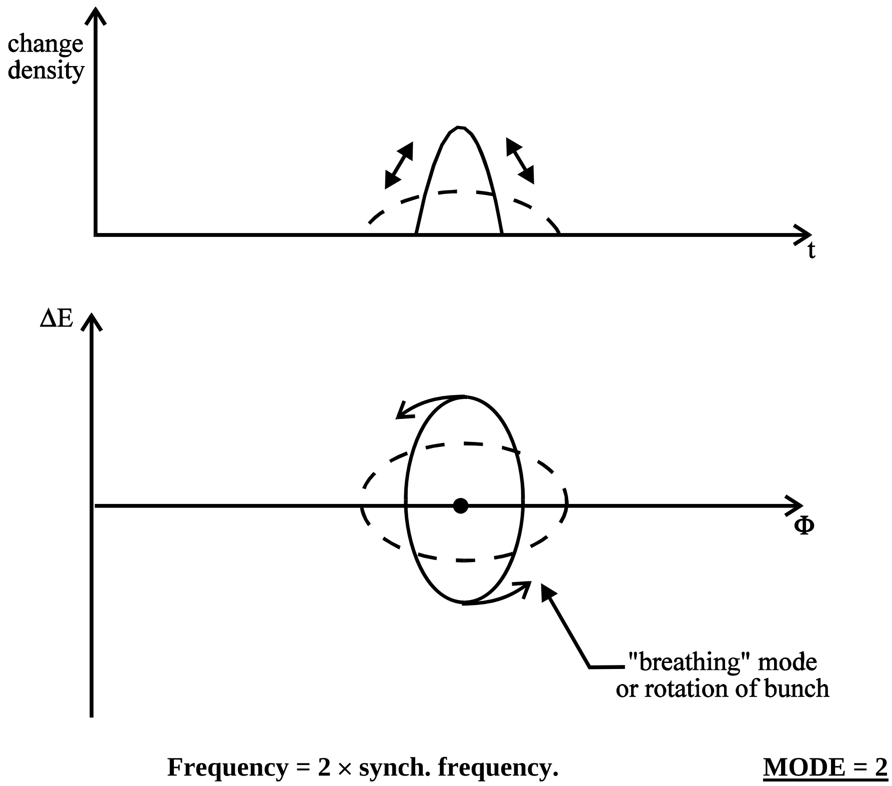


Figure 7: Mode 2 or quadrupolar longitudinal instability

Figure 8 shows two real cases taken from the CERN PS. (just to prove these things really exist!)

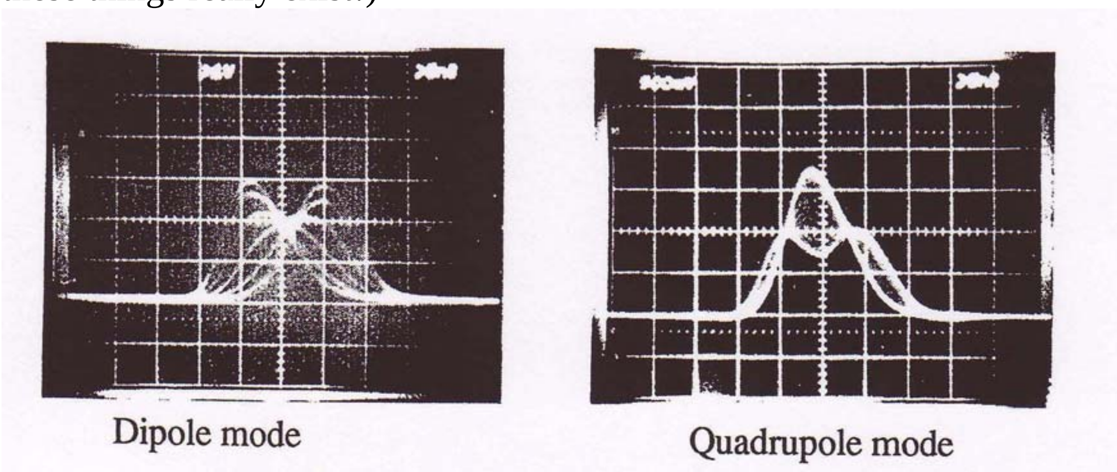


Figure 8: Single bunch dipole and quadrupolar mode oscillations from real-life

It is possible, indeed, probable, that the field induced in a cavity will remain behind long after the bunch has passed, and will therefore, affect any following bunches. This will lead to multi-bunch instabilities, known as coupled bunch modes.

The induced field due to the first bunch passing through a cavity drives the motion of second bunch, which in turn excites the third bunch etc. Until the first bunch appears back at the cavity for a second time, and the process continues.
For example for 4 bunches we get 4 possible modes of coupled bunch oscillation, see figure 9.

M=0: all bunches oscillate in phase.

M=1: $\frac{\pi}{2}$ between phase of the oscillation of each bunch.

M=2: π between phase of the oscillation of each bunch.

M=3: $\frac{3\pi}{2}$ between phase of the oscillation of each bunch.

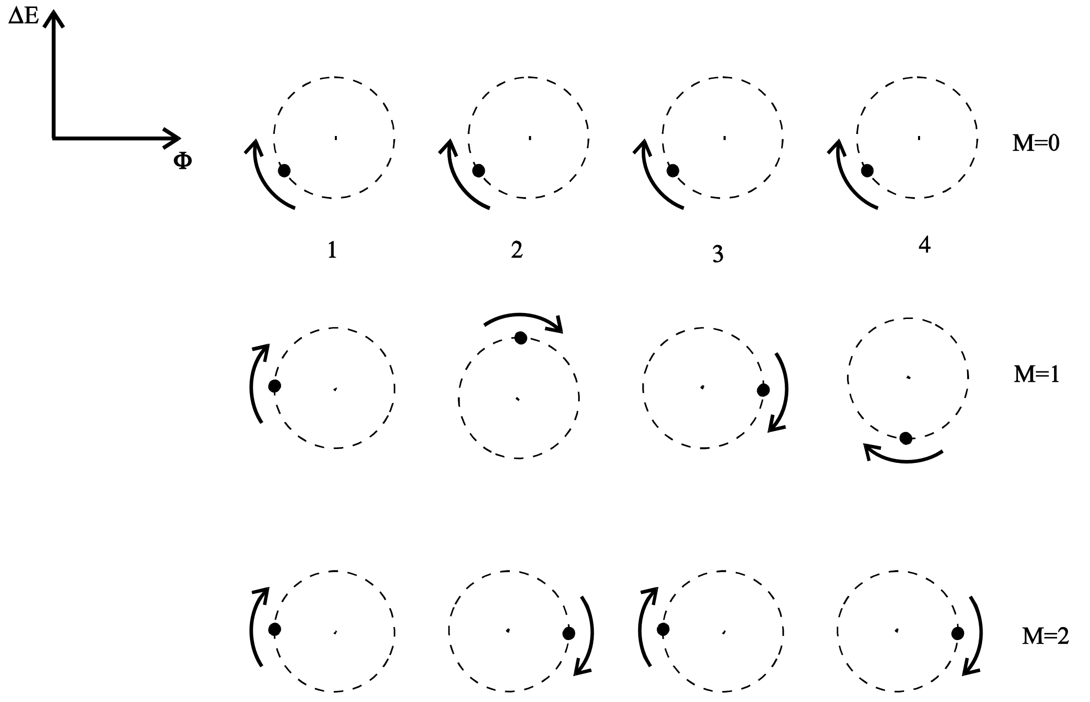


Figure 9: Different modes of coupled bunch oscillations for 4 bunches

The important question is, are these modes stable? I.e. do they damp or amplify themselves?
Let us start by looking at the mode **M=1** . Initially consider the case where each bunch is stable i.e. there are no bunch oscillations. Now put a cavity somewhere in the ring, which resonates at the bunch revolution frequency. Each bunch will induce a voltage in this cavity, which will affect all four bunches. This is shown in figure 10.

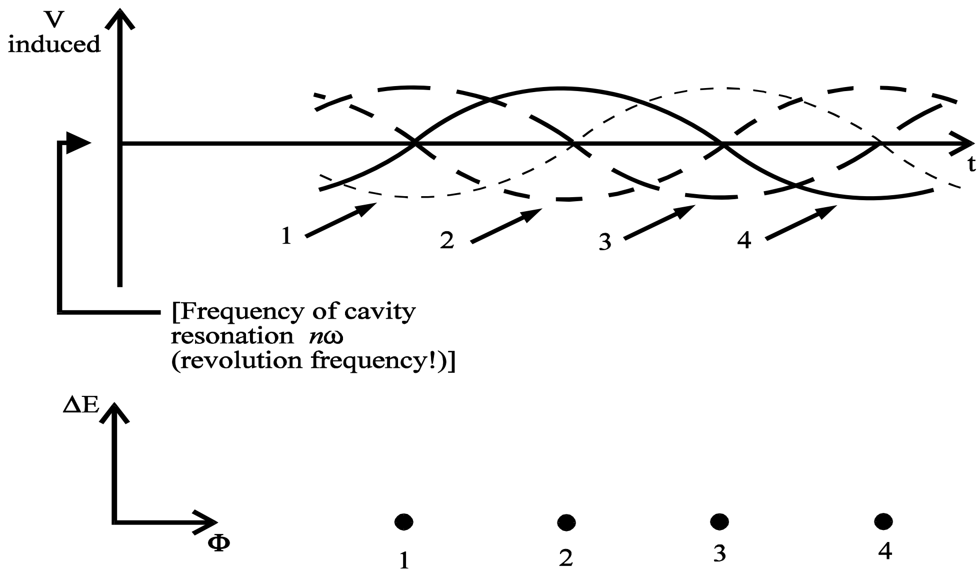


Figure 10: Voltage induced in a cavity by 4 equi-distant bunches

It is not too hard to see that the overall effect will be zero as the voltage induced by bunches 2 and 4 will exactly cancel each other, and likewise for 1 and 3. Therefore there will be no net effect on the individual bunches.

Now set up a coupled bunch oscillation, so that each bunch executes phase oscillations, with a phase difference of $\frac{\pi}{2}$ (mode M=1) between each bunch. This is shown in figure 11.

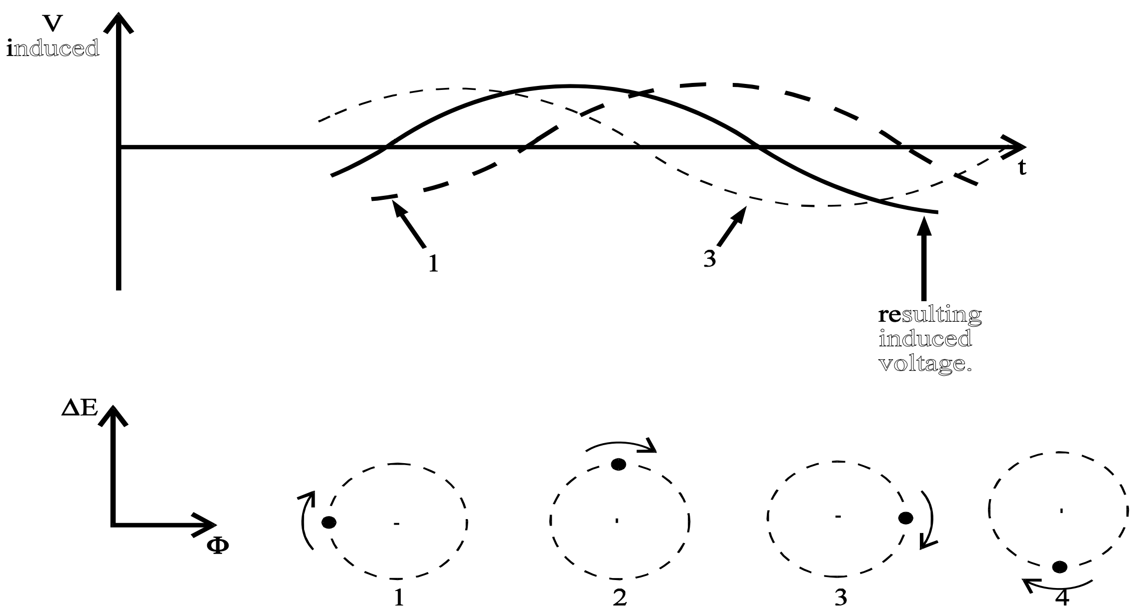


Figure 11: Voltage induced in a cavity during M=1 coupled bunch oscillations

Exactly as before the voltage due to bunches 2 and 4 cancel, but 1 and 3 are displaced and no longer exactly cancel, therefore there is a net residual voltage in the cavity. Bunches 1 and 3 are relatively unaffected, but bunch 2, which has a positive energy error, sees an accelerating voltage. therefore it's energy error is increased. Bunch 4 has a negative energy error and sees a decelerating voltage. This is already not a very good sign.
Now let each particle go through 1/4 of a phase oscillation. See figure 12.

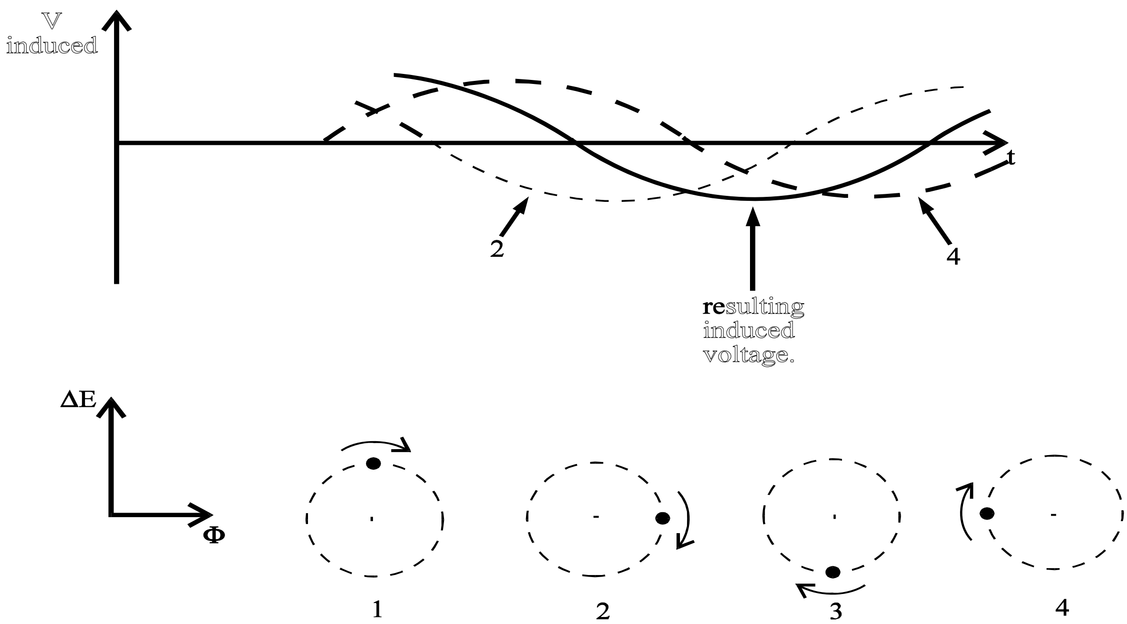


Figure 12: Voltage induced in a cavity during M=1 coupled bunch oscillations ¼ of synchrotron period later

Now the voltage induced by 1 and 3 cancel and 2 and 4 are displaced. If we follow the same arguments as before we see that the energy displacements of bunches 1 and 3 are increased by the residual voltage left in the cavity.
Therefore we can conclude that, under theses conditions, mode M=1 is **unstable.**
If we repeat the exercise for mode M=3, we see that it is stable. This exercise is shown in figure 13. The energy of bunch 1 is too high and it is decelerated. The energy of bunch 3 is too low and it is accelerated. This leads directly to a damping of the energy oscillations for all bunches in mode M=3.

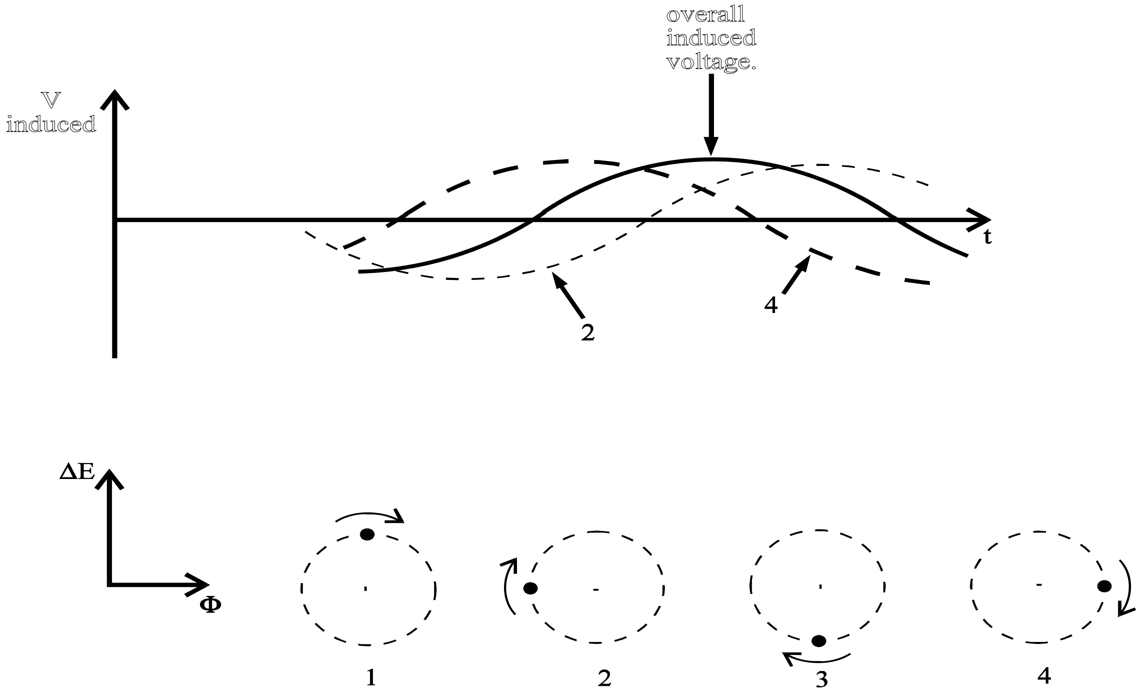


Figure 13: Coupled bunch oscillation mode M=3

Possible Cures:

Single bunch modes.

1) Robinson Instability; tune RF cavities so that ω_{RF} (resonant frequency) $\langle h\omega$.
2) Match the RF bucket to the bunch to avoid single bunch M=2 modes.
3) Use a phase lock system, which measures the phase difference between bunch and RF and then modulates the RF frequency to damp any energy oscillations (M=1 mode).
4) Radiation damping will tend to damp energy oscillations provided the damping time is less than the instability growth time.

Multi-bunch modes.

1) Reduce the longitudinal impedance of the accelerator as much as possible, which means limiting changes in vacuum chamber shape and cross section, and making smooth transitions from one section to the next.
2) Use feedback systems, which measure the longitudinal position of each bunch and correct them with a high frequency accelerating cavity.
3) Radiation damping, again this can only help as with single bunch modes
4) Damp resonant modes in cavities. By using damping antennas in cavities to absorb the energy left behind in the cavity after the passage of a bunch, the voltage seen by successive bunches can be greatly reduced.

Bunch lengthening.

Assuming we have cured single bunch instabilities, and controlled the coupled bunch modes. There is a strong possibility that we will still see a problem, in that when we increase the bunch current, the peak bunch current will not increase beyond a certain level. I.e. the bunch gets larger and longer as we add more current. This is a serious problem in colliding beam machines, where short bunches and high peak currents are important.

In general the overall Broad-Band impedance of a machine will be inductive. The Broad-Band impedance is the longitudinal impedance of the rest of the machine not including the high-Q resonant cavities. As a result of this inductive impedance the induced voltage will tend to lag behind the bunch. See figure 14.

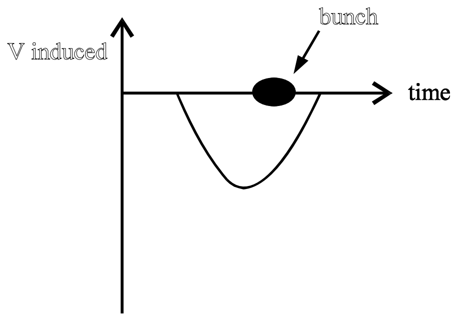


Figure 14: Induced voltage for an inductive impedance

If we add this to the RF voltage on the RF cavity, which determines the initial bunch length we find that the bunch sees a reduced voltage due to this inductive impedance, and this reduced slope will tend to lengthen the bunch. See figure 15.
As we increase the bunch intensity the induced voltage increases, and the voltage slope seen by the bunch decrease further, causing the bunch to get progressively longer. Increasing the RF voltage will have the same effect as this will also try to increase the bunch peak current, which will in turn increase the induced voltage and reduce the overall voltage seen by the bunch.

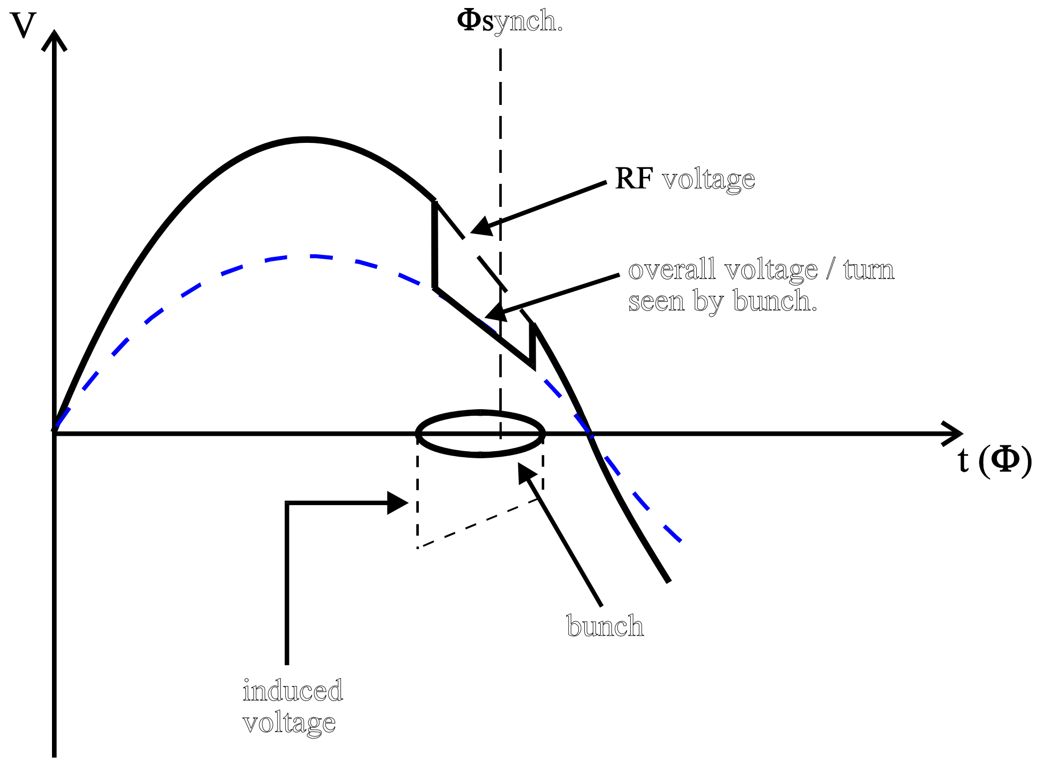


Figure 15: A schematic view of bunch lengthening

Chapter 10: Transverse Beam Instabilities

Here we will follow very much the same format as in the previous chapter on Longitudinal Instabilities. We saw in chapter 9 that the charged bunches will induce a charge on the inside of the vacuum chamber, which leads to an image current on the vacuum chamber walls. These image currents will not only affect the longitudinal motion of the bunch, but also it’s transverse motion.

For a single bunch circulating in a storage ring, if the whole bunch is displaced transversely it will oscillate around some mean position. (horizontal or vertical betatron oscillations). This oscillation will drive a differential wall current, i.e. when the bunch is closer to one side of the vacuum chamber than the other it will induce a larger wall current on this side. This differential wall current leads to a residual magnetic field in the centre of the vacuum chamber, which will deflect the moving, charged bunch. See figure 1.

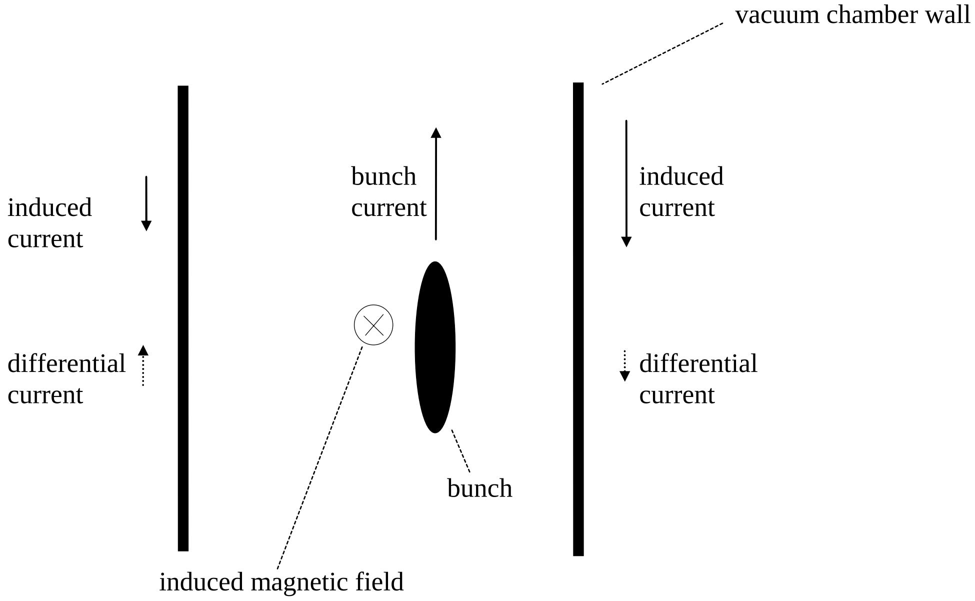


Figure 1: Induced wall currents leading to the transverse deflection of a moving bunch.

As in the longitudinal case we characterise the electromagnetic response of the beam environment to the beam as a “transverse coupling impedance”:

$$\int_0^{2\pi r} \left(\mathrm{E} + \mathrm{v} \wedge \mathrm{B} \right) \mathrm{ds} = \int \left(\mathrm{I}(\omega) \times \mathrm{Z}\perp(\omega) \right) \mathrm{d}\omega$$

induced transverse E and B fields summed around the ring

frequency spectrum of bunch current

transverse impedance

$Z_\perp$ is itself a function of frequency, which means that there will very important differences in the behaviour of long and short bunches in the same storage ring. As for the longitudinal impedance, there will be resistive, capacitive and inductive components for the transverse impedance of the ring. There is, however, one very important difference between longitudinal and transverse impedance. For a vacuum chamber with a short non-conducting section, the main image current sees a very high impedance, which leads to a large longitudinal impedance (e.g. an RF cavity). This non-conducting section does not greatly affect the differential image current loops in the vacuum chamber, and, so this short non-conducting section has a much smaller effect on the overall transverse impedance of the ring.
To conclude, any structure that will support current loops in the beam surroundings will increase the transverse impedance, and interruptions to the smooth vacuum chamber increase the longitudinal impedance.

In chapter 9, on longitudinal instabilities, we only dealt with synchrotron oscillations. Transverse instabilities are more complicated as we must consider both betatron and synchrotron oscillations simultaneously. The synchrotron, or energy, oscillations are important here because as the particles move around the ring they undergo synchrotron oscillations, which modify their energy. If the chomaticity is not exactly zero, then as a particle's energy changes then its Q value, or number of betatron oscillations/turn, will also change. This effect is very important for transverse instabilities.

Consider a single bunch oscillating transversely in a ring with zero chromaticity. Therefore all the particles will have the same betatron frequency regardless of their energy. Their will be various modes of oscillation possible:

Rigid bunch mode (M=0)
The bunch moves transversely as a rigid unit. So on a transverse position monitor we would see the centre of gravity of the beam oscillating backwards and forwards at the betatron frequency. See figure 2. The change in radial position on each successive turn will be given by the betatron phase advance/turn.

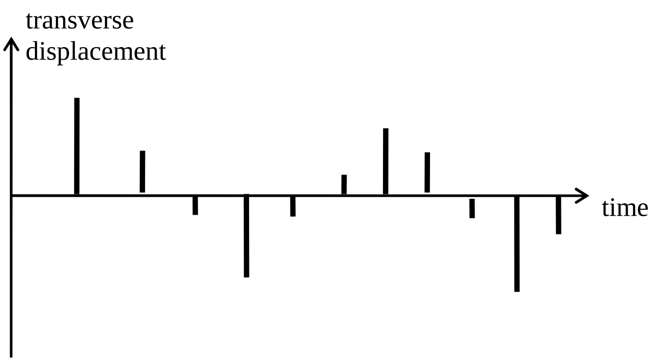


Figure 2: Position signal from a beam position monitor for a transverse bunch oscillation

If we record this signal over many turns and superimpose each consecutive turn, we see a standing wave with a single node. See figure 3.

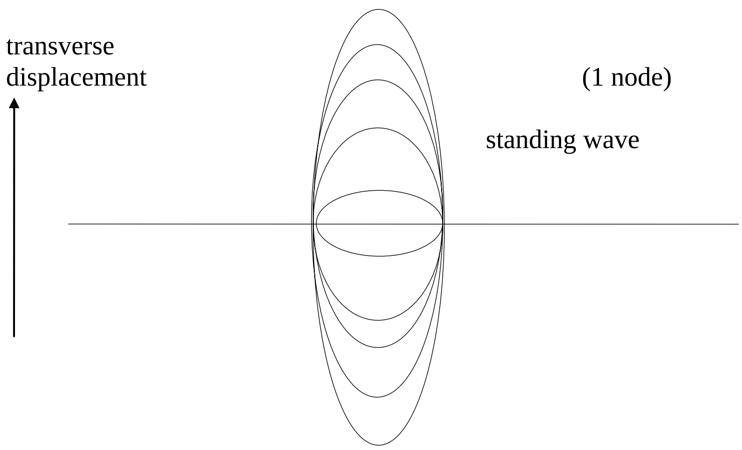


Figure 3: Mode M=0 transverse bunch oscillation

There are higher order modes of transverse bunch oscillation, and these are called “head-tail” modes, because the electromagnetic field, induced by the passage of the head of the bunch, excites oscillations in the tail of the bunch. In such higher order mode oscillations the centre of gravity of the bunch may not move, which makes them more difficult to detect. The mode (M=1) head-tail oscillation is shown in figure 4. Here the head and tail of the bunch move 180° out of phase with each other, i.e. when the head of the bunch moves upwards the tail moves downwards. In this case the centre of gravity of the bunch does not move. Figure 5 shows an M=2 mode.

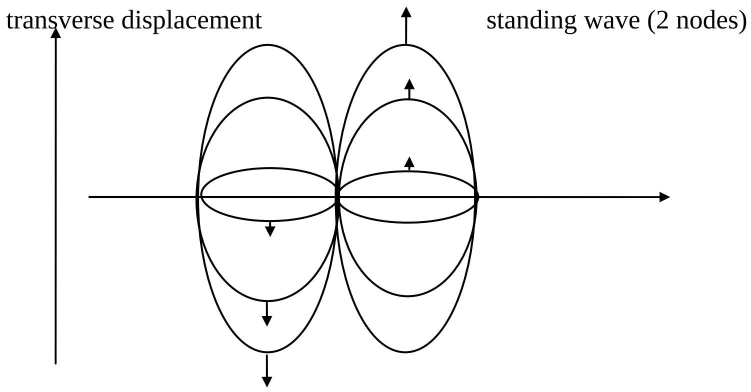


Figure 4: Mode (M=1) Head Tail instability

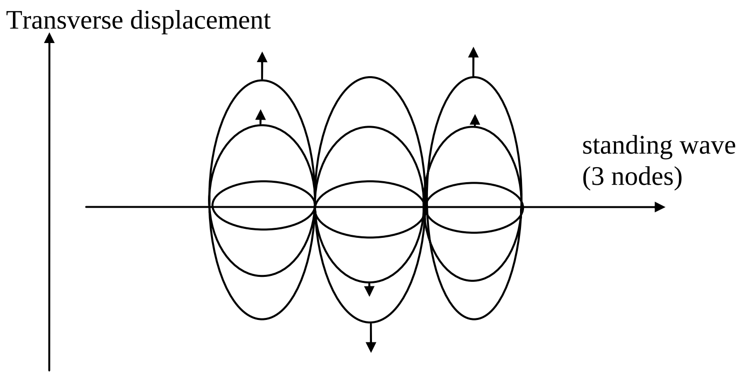


Figure 5: Mode (M=2) Head Tail instability

The next step is to look at the conditions under which these transverse oscillations will be stable or unstable.

In order to drive any oscillatory system to ever increasing amplitude (instability) the driving force must be ahead, in phase, of the oscillatory motion. If the driving force is exactly in phase or 180° out of phase with the motion there is no net effect. If the driving force lags behind the motion it will actually reduce the oscillation amplitude. If you have small children, try this next time you push them on a swing!

Now we must consider what happens when we set up a transverse oscillation, but remembering that the particles undergo betatron and synchrotron oscillations simultaneously. For a M=1 mode oscillation, this is shown in figure 6. Figure 6 is a longitudinal phase space plot on which the particles rotate as they undergo synchrotron oscillations. However we have also set up a head tail motion whereby as the head of the bunch moves up the tail moves down. This is shown by the +’s and -’s in Figure 6. There will of course be many betatron oscillations for one synchrotron oscillation, but after one half of a synchrotron oscillation period the head and the tail of the bunch will have been exchanged.

Under these conditions, for zero chromaticity, the head an the tail of the bunch are always exactly 180° out of phase and therefore the driving force, which is generated by the wake field of the head of the bunch will not drive the tail into instability.

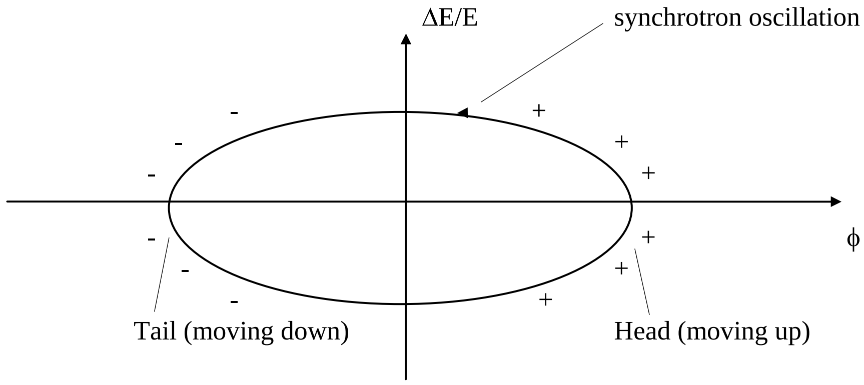


Figure 6: Synchrotron and betatron motion during head tail (M=1) oscillation

Unfortunately there may well be some chromaticity in the machine. Without sextupoles the chromaticity is naturally negative. I.e. the Q decreases as the particle energy increases. In figure 6 when a particle moves from the head to the tail of the bunch it's energy increases therefore the number of betatron oscillations it has undergone will be slightly less if the chromaticity is negative. Therefore the phase of the betatron oscillation at the tail of the bunch will tend to lag slightly behind the head of the bunch. This means that the driving force generated by the wake field left behind by the head of the bunch will be ahead, in phase, of the tail. This leads directly to unstable motion!

For a positive chromaticity, and following the same argument, we can see that the head tail instability will be damped as the driving force created by the wake field of the head of the bunch will be behind, in phase, the motion of the tail.

In conclusion, to damp this kind of head-tail instability we must ensure that the chromaticity is positive, therefore we will need sextupoles as most accelerators have a naturally, i.e. without sextupoles, negative chromaticity. Figure 7 shows examples of various modes of transverse instability as observed in the CERN PS and PSB.

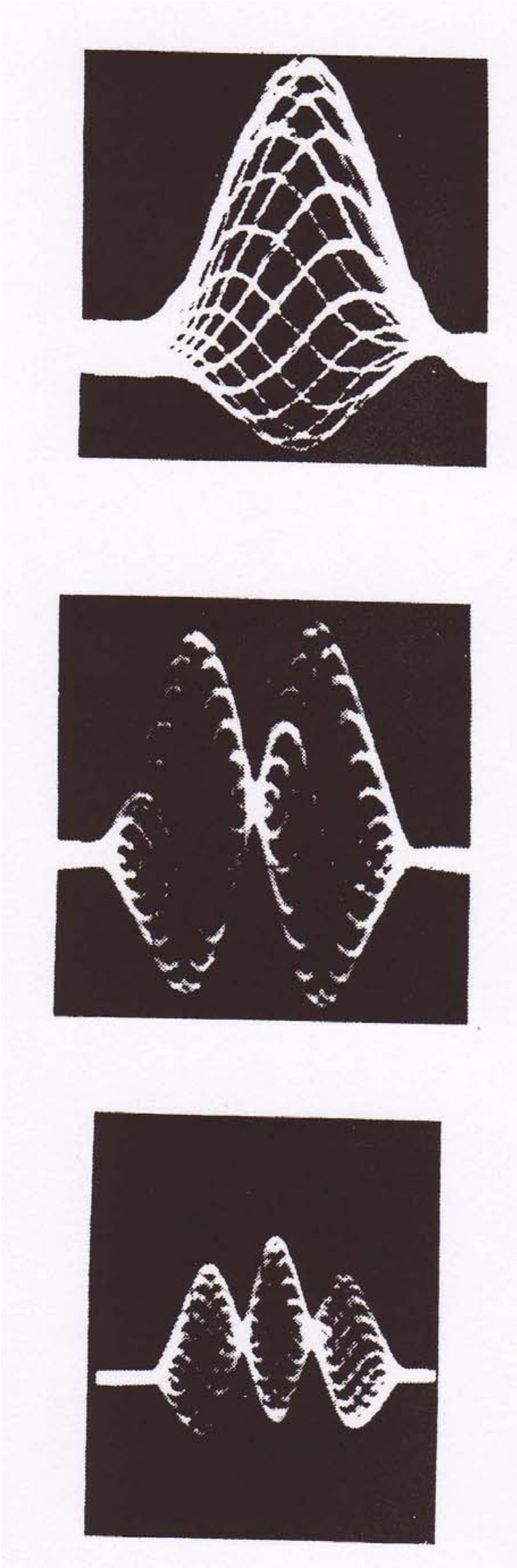


Figure 7: Mode M = 0, 1 and 2 transverse instabilities from the CERN PS and PSB

Multi-bunch transverse instabilities are also possible, and are similar to longitudinal multi-bunch effects, where some transverse component of the induced electro-magnetic field remains in a cavity until the next bunch arrives. However they are rarely observed as the thresholds for longitudinal multi-bunch instabilities are much lower.

Cures for transverse instabilities.

1/ Correct the natural chromaticity of the accelerator with sextupoles, However we should be careful as if we set exactly zero chromaticity then the rigid bunch node oscillation is very easy to excite, as all the particle have exactly the same transverse oscillation frequency.

2/ Damp transverse modes induced in the cavities with absorbing antenna

3/ Use a transverse feedback system, which detects the coherent transverse motion of a bunch and applies a transverse kick to damp it. This is only appilicable if the centre of gravity of the bunch moves, i.e. for the M=0 “rigid bunch” mode.

4/ Install Octupole magnets, which will introduce a variation in betatron tune with oscillation amplitude. Then the rigid bunch mode oscillation will be damped as any increase in oscillation amplitude will change the betatron tune and the coherent particle motion will rapidly decohere.

The addition of sextupoles and octupoles will, however, tend to excite 3rd and 4th order resonances and will restrict the choice of betatron tune for the accelerator.

Space Charge de-tuning (or Laslett tune shift)

The beam in our accelerator is made up of many charged particles, and each particle is electrostatically repelled by the rest of the beam. However, the individual particles each behave as a current, due to their motion, and like currents attract. At low momenta ($\beta<1$) the electrostatic repulsion dominates and as $\beta \Rightarrow 1$, then the two effects cancel. This gives rise to a defocusing effect, at low momenta, in both transverse planes, which reduces Qh and Qv. The change in Q, ΔQ, is given by:

$$\Delta Q = \frac{Z^2}{A} \times N \times \frac{1}{\varepsilon} \times \frac{1}{B_f} \times \frac{1}{\beta^2} \times \frac{1}{\gamma^3}$$

This assumes the beam is a uniform cylinder of charge
N = number of particles, ε = transverse beam emittance, β, γ = relativistic parameters and B_f = Bunching factor (1 for a coasting beam, < 1 for a bunched beam, << 1 for a beam of very short bunches)

So ΔQ is large for high intensity (N), small transverse emittances (ε), short bunches (B_f), and low momenta ($\beta^2\gamma^2$). For a coasting beam with a realistic density distribution this is only really correct for small amplitude betatron oscillations, in which the particles only see the central dense part of the transverse beam distribution. For large amplitude betatron oscillations, the particle spends time out in the low density tails of the beam and so the ΔQ will be less important. This leads to a tune spread of approximately $\Delta Q/2$ in the beam. Such a spread in tune can mean that particles will cross unwanted betatron resonances and may get lost out of the storage ring. For bunched beams, the particles also execute synchrotron oscillations, which are very slow compared with betatron oscillations. In the course of a synchrotron oscillation a particle from the head of the bunch will travel from the low density “head”, through the high density centre of the bunch out to the low density “tail”. This means that the betatron tune will be modified by the changing space charge effects at the synchrotron frequency. In high intensity, low energy machines this is a very strong effect.
Depending on its design and energy etc. any accelerator can support a certain tune spread before particles start being lost.

- Fast cycling synchrotrons, where the beam is rapidly accelerated and ejected, can support $\Delta Q \rightarrow 0.3$ or more
- Storage rings in which the beam circulates for many hours or days, can only allow $\Delta Q \rightarrow 0.01$ to 0.02

As examples, these effects are very important in the CERN PS Booster, at very high beam intensity, and LEAR at very low beam energies. For LEAR at 61 MeV/c, if the maximum sustainable ΔQ is 0.02, which is already very large. Then the maximum bunched beam intensity is $1.2\ 10^9$ particles.
For other accelerators, however, the effect is completely negligible. In the CERN Antiproton Accumulator at 3.5 GeV/c, for stack of $1\ 10^{12}$ particles, the space charge tune spread is < 0.001.

Chapter 11: Schottky Noise and Stochastic cooling.

Schottky noise - Longitudinal plane.

For a single particle circulating in a storage ring, with a revolution frequency given by $f_0 = \frac{1}{T} = \frac{\omega}{2\pi}$. The time dependent signal seen on a longitudinal pick-up will be:-

$$i(t) = e \sum_{n=-\infty}^{\infty} \delta(t - nT)$$ where e is the particle charge

This will be the response seen on an oscilloscope, for a single particle. The frequency response will be given by the Fourrier transformation of this time dependent signal:-

$$i(\omega) = ef_0 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega t} \delta(t - nT) dt$$

$$i(\omega) = ef_0 \sum_{n=-\infty}^{\infty} e^{i\omega nT}$$

$$\therefore i(\omega) = ef_0 + 2ef_0 \sum_{n=1}^{\infty} \cos(n\omega T)$$

ef_0: DC term (beam current).

$\sum_{n=1}^{\infty}$: successive harmonic of f_0.

This frequency response will consist of a series of frequencies at exact harmonics of the revolution frequency. This will be the response that would be seen on a spectrum analyser. See figure 1.

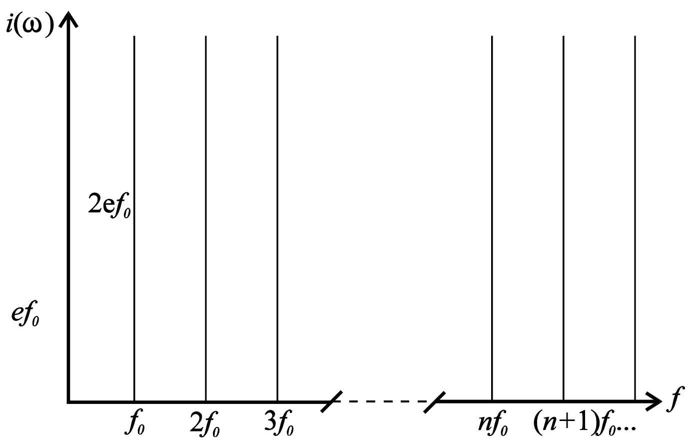


Figure 1: Frequency response for a single circulating particle

If we now add many particles to form a coasting beam with no large-scale longitudinal structure, then, in the time domain on the oscilloscope, there will be no net response. This is because each particle has a random initial longitudinal phase. However in the frequency analysis each particle now contributes it's own series of harmonics for it's own revolution frequency f_i, where $\left[f_i = \frac{1}{T_1} = \frac{\omega_1}{2\pi} \right]$. These harmonics are contained within a narrow band of frequencies around the central revolution frequency, f_0, where the width of each band is given by:-

$$\Delta f = n f_0 \eta \Delta p/p \qquad \text{where } \eta = \left[\frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} \right].$$

Provided $\Delta f \langle\langle f_0$, we now sum over N particles to obtain the overall frequency response of the beam:

$$I(\omega) = Nef_0 + 2ef_0 \sum_{k=1}^{N} \sum_{n=1}^{\infty} \cos\left(n\omega_k T_k + \phi_k\right)$$

For a coasting beam ϕ_k the initial longitudinal phase is random.

$$\therefore \langle I \rangle = Nef_0 \quad \text{for } n = 0 \left[\text{DC beam current} \right]$$

$$\text{But } \langle I \rangle = 0 \quad \text{for } n \geq 1$$

However the r.m.s. current/band is non-zero, and for any one band, i.e. a particular value of *n*:-

$$\left\langle I^2 \right\rangle = 4e^2 f_0^2 \frac{N}{2}$$

$$\therefore I_{rms} = 2ef_0 \sqrt{\frac{N}{2}}$$

It is important to note that the r.m.s. current/band is independent of the band chosen, or *n* the harmonic number. This r.m.s. current is the so called Schottky noise current and can be observed by tuning a spectrum analyser to the frequency of the band under consideration. These bands are schematically shown in figure 2.
The power in the schottky band is proportional to the number of circulating particles and the revolution frequency squared. Therefore this noise power drops off rapidly for low intensity beams at low energies.

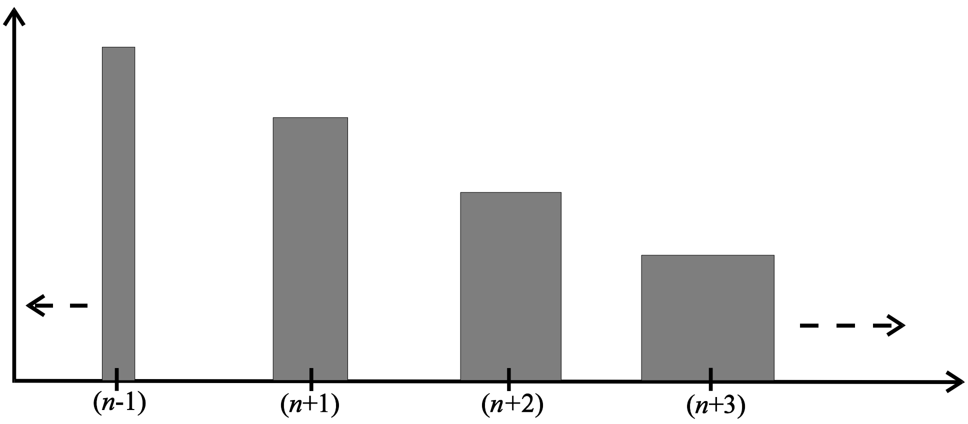


Figure 2: Longitudinal schottky noise bands

Remember $\dfrac{\Delta f}{nf_0} = \eta \dfrac{\Delta p}{p}$

$$\therefore \frac{\Delta p}{p} = \frac{\Delta f}{nf_0 \eta}^{\ *}$$

*This is true provided *n* is small enough that the bands do not overlap.

Therefore the power in the schottky band, measured as a function of revolution frequency, is proportional to the momentum distribution of the particles in the beam. This is important as it allows us to observe directly, and non-destructively, the momentum distribution of particles in the beam.

Schottky noise - Transverse Plane.

If the pick-up used to observe the beam is made sensitive to transverse beam position. Then, for a single particle, betatron oscillations will be observed. If a = amplitude of the oscillation and q = fractional betatron tune, then the time dependent signal for a single particle becomes:-

$$i(t) = ea\cos(q\omega t)\sum_{n=-\infty}^{\infty}\delta(t - nT)$$

Exactly as before in the frequency domain:-

$$\mathrm{i}(\omega) = \mathrm{ef}_0 \sum_{\mathrm{n}=-\infty}^{\infty} \cos \mathrm{q}\omega\mathrm{T}\mathrm{e}^{\mathrm{in}\omega\mathrm{T}}$$

For one single value of *n*:-

$$i(\omega) = 2ef_0 a \cos(q\omega T)\cos(n\omega T)$$

$$i(\omega) = ef_0 a\left[\cos(n+q)\omega T + \cos(n-q)\omega T\right]$$

Therefore each longitudinal schottky line for a single particle develops two sidebands. This is exactly the same as for Amplitude Modulated (AM) radio transmission. Here the "carrier frequency" is the revolution frequency and the betatron oscillation provides the amplitude modulation. Again we now sum over all N particles in the beam, each having their own f_k, a_k, q_k and a random initial betatron phase ϕ_k. So for one value of *n*:-

$$i(\omega) = 2ef_0 \sum_{k=1}^{N} ak \cos(n \pm qk)\ \omega_k T_k + \phi_k$$

As for the longitudinal case:-

$$\left\langle i_{(\omega)} \right\rangle = 0$$

$$\left\langle i_{(\omega)^2} \right\rangle = 4e^2 f_0^2 \left\langle a^2 \right\rangle \tfrac{N}{2}$$

$$i_{rms} \text{ / band} = 2ef_0 a_{rms} \sqrt{\frac{N}{2}}$$

This is the so-called transverse schottky noise signal and can be observed by tuning a spectrum analyser to the output of a transverse position sensitive pick-up. Again $i_{r.m.s.}$ is independent of *n*. But the transverse oscillation amplitude of the particles in the beam is directly related to the transverse beam emittance:-

$$a_{rms} \propto \sqrt{\varepsilon}$$

Beam emittance.

R.m.s. oscillation amplitude of the particles.

Therefore, the total power in one transverse sideband is proportional to εN. But the total power in one longitudinal band is proportional to N.

Therefore the transverse emittance can be derived by dividing the total transverse schottky noise power by the total longitudinal schottky noise power.

If we measure the frequencies of two adjacent bands f_+ and f_- then :-

$$f_+ = (n+q)\, f_0;\; f_- = (n-q)\, f_0$$

$$\therefore q = \frac{(f_+ - f_-)}{2\, f_0}$$

In this way we can measure the transverse beam emittance and the fractional part of the betatron tune, q. Here however, care must be taken to ensure that the measured points correspond to the same particles in the beam. This method is used for tune measurement in the CERN Antiproton accumulator.
At the CERN Low Energy Antiproton Ring a slightly different technique is used to measure the tune of the low intensity coasting beam. See Figure 3.

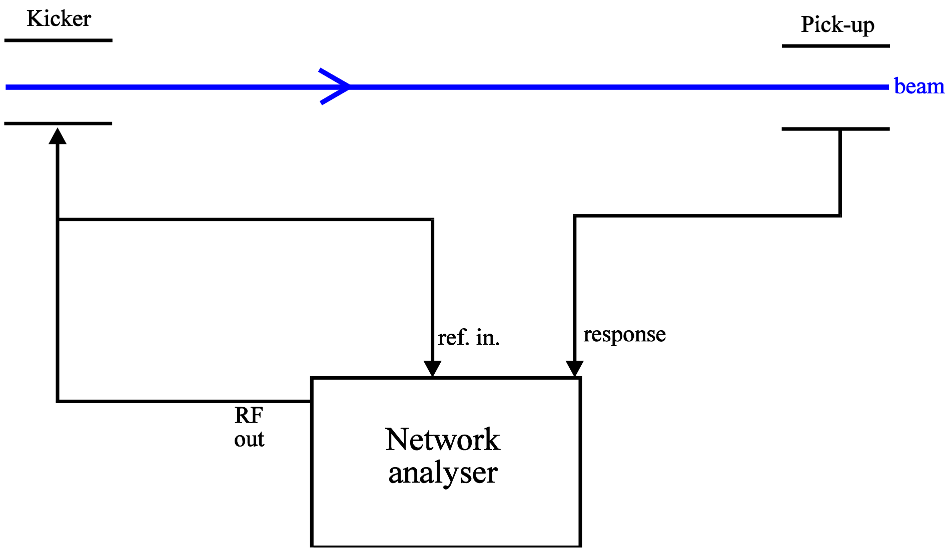


Figure 3: Beam transfer function measurement

The idea is to excite the beam at the frequency close to one of the betatron sidebands. The network analyser is used to give the amplitude and phase response of the beam to this excitation. The response is very strong when the excitation frequency corresponds exactly to one of the betatron sidebands. See figure 4.

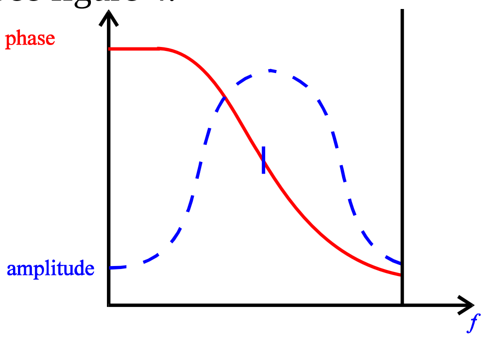


Figure 4: Amplitude and phase response of the beam as a function of frequency to a transverse excitation of a betatron sideband.

This is known as a beam transfer function (BTF) measurement and the coherent response of the beam and gives much bigger signals than the incoherent schottky noise. The BTF is very useful for measuring the betatron tunes for very low intensity beams, and it also gives information about the response of the beam to its surroundings. See chapter 10 on transverse impedance.

The width, Δf, of a transverse schottky side band depends on η, and ξ, the chromaticity, where:-

$$\eta = \frac{\Delta f / f}{\Delta p / p}$$

$$\xi = \frac{\Delta Q / Q}{\Delta p / p}$$

$$\therefore \Delta f = \left[(n \pm q)\eta \pm Q\xi \right] \Delta p / p . f$$

(remember q = fractional part of Q).

Therefore by measuring the widths of pair of sidebands we can measure the chromaticity, ξ. This is usually only true for small values of n, since, for large values of n, the change in line width caused by the chromaticity is too small to measure.

In conclusion we can use Schottky noise to:

1) Measure longitudinal and transverse beam emittances
2) Measure betatron tune
3) Measure chromaticity.

In addition BTF measurements will give us information about beam stability.

Stochastic Cooling, a brief introduction

The physical beam size is determined by the longitudinal and transverse beam emittance. Stochastic cooling aims at reducing these emittances and hence reducing the physical beam dimensions. This implies an increase in the phase space density. This is useful for:-

1) Accumulation of rare particles ($\bar{p}$'s in AAC).
2) Restoring beam quality after blow-up, due to instability, deceleration... (LEAR)
3) Increasing interaction rates and energy resolution in hadron colliders [it is not needed in electron/positron machines!!].

Firstly we will consider transverse cooling. Imagine one particle in a storage ring. This particle is oscillating due to a transverse injection error. In this storage ring we have installed a transverse beam cooling system, which consists of a position sensitive pick-up, to detect the displacement of the particle, and a transverse kicker, to apply a correction. See figure 5.

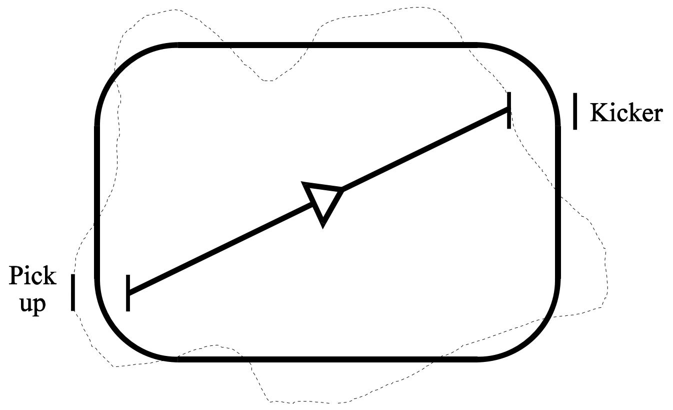


Figure 5: A single particle and a schematic transverse cooling system.

The Pick-up will detect a position error, and the kicker will only correct an angle error. Therefore, if the pick-up detects a position error, as a result of the large transverse oscillation, we must position the kicker so that it applies a kick to cancel this oscillation. This will reduce the oscillation amplitude and "cool" the beam.

This means that the betatron phase advance from the pick-up to the kicker must be well chosen. See figure 6.

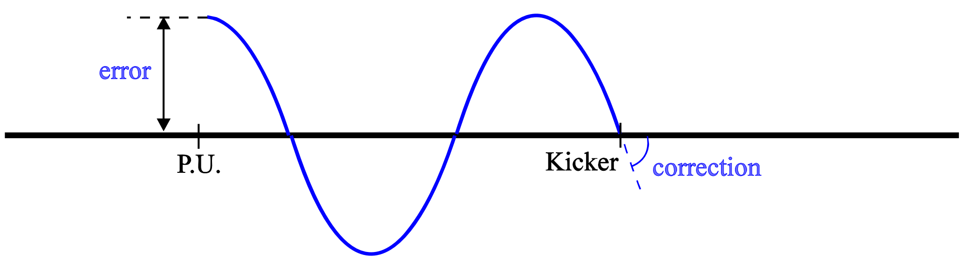


Figure 6: Optimum phase advance from pick-up to kicker

Ideally the phase advance of from Pick-up to kicker should be:

$$= \left(\frac{2n+1}{2}\right)\pi$$

This means that the positions of the elements of the transverse cooling system are determined by magnetic lattice. [This is not the case for longitudinal cooling as we will see later].

In practice we must add a lot of signal amplification as the detected signals are very small. So the pick-up detects a transverse position error and this signal is amplified and fed to a kicker, which applies the transverse kick to the particle.

This very basic cooling system works for a single particle, but our beam contains many particles, and the cooling system amplifier will have a limited bandwidth. Therefore it will not be possible to detect and correct individual particles in the beam.

Sampling theory tells us that, when an infinitely short pulse passes through a filter of bandwidth W, the resulting pulse has a width of 1/2W seconds.

Therefore one individual particle will experience the "correcting kicks" due to all the particles which pass through the pick-up within a time ±1/4W. Where W is the overall system bandwidth.

For a uniformly distributed, coasting beam of N particles, with a revolution time T, then each particle will experience it's own correction kick and the kicks due to N/2WT neighbouring particles.

These N/2WT particles are called a beam sample (N_s), and the number of particles in a beam sample (N_s) is determined by the beam intensity and the bandwidth of the cooling system hardware.

E.g. in LEAR at 609 MeV/c: N = 1 10^9, T = 500 nsecs, W = 500 MHz

Then the number of samples/turn = 500

Number of particles in a sample = 2 10^6

Now let our original particle receive a correction over one turn, Δx:

$\Delta x = -\lambda x$ λ is determined by the overall gain of the system

and x = measured error

This test particle will also experience the correction kicks due to all the other particles in the beam sample:

$$\Delta x = -\lambda x - \lambda \sum_i x_i$$

The first term is the coherent cooling term and the second term is an incoherent term, which is summed over the rest of the beam sample.
Now ,we can define the centre of gravity of our beam sample as:

$$\langle x_s \rangle = \frac{1}{N_s} \sum_{i=1}^{N} x_i$$

Rewriting the expression for the correction applied to our test particle:

$$\Delta x = -\lambda \sum_{i=1}^{N} x_i$$
$$\therefore \Delta x = -\lambda N_s \langle x_s \rangle \quad = -g \langle x_s \rangle$$

The quantity λN_s is called the gain of the system, and it depends on the hardware gain, system bandwidth and beam intensity. Therefore the correction seen by a single particle on a single turn is the correction of the centre of gravity error of the sample containing that particle. This is often called the observed sample error.

To obtain a very rough estimate of the cooling rate in such a system, let us ignore the incoherent term, and assume that the overall system gain, g, is unity. I.e. we do not apply a larger correction than the observed sample error.
NB. This assumption about the correction not being larger than the observed error means that we must reduce the hardware system gain, λ, if we increase the number of particle, N_s.

$$g = 1 = \lambda N_s$$
$$\therefore \Delta x = -\lambda x = -(1/N_s)x$$
$$\therefore \Delta x/x = -(1/N_s)$$

The reduction in x over a number of turns, n, is an exponential decay of the form:

$$x = x_0 \exp(-n/\tau_n)$$
$$\therefore \frac{dx}{dn} = -\frac{x}{\tau_n}$$

However, for a single turn we can set, dn = 1.

$$\therefore \frac{1}{\tau_n} = -\frac{dx}{x} = \frac{1}{N_S}$$

In this case the cooling rate per second is simply $(1/\tau_n)*f_0$, where f_0 is the revolution frequency. But remember that $N_s = N/2WT_0$, where T_0 is the revolution time $(1/ f_0)$. Now the cooling rate per second is:

$$\frac{1}{\tau} = \frac{2W}{N}$$

This demonstrates that to get fast cooling we need a large system bandwidth and a small number of particles in the beam.

In order to make this very simplified estimation we have neglected the incoherent response due to all the other particles in the sample and amplifier noise. This second effect is very important as we need very high gain amplification systems. If we include both these effects, our cooling time becomes:

$$\frac{1}{\tau} = \frac{2W}{N}\left[2g - g^2(1+U)\right]$$

Where U is the ratio of the noise from external sources to the observed sample error. Even if U is large, i.e. a very poor signal to noise ratio, cooling is still possible provided we choose g (or λ) small enough. So, if we have a noisy system, we have to reduce the amplifier gain so that:

$$2g > g^2(1+U)$$

Initially we made an assumption that all our beam samples are random. Now imagine that all the particles have the same revolution frequency. All cooling action will stop after a single turn, as the centre of gravity error of every sample in the beam will now be zero! Fortunately this is not the case and faster particles overtake slower particles and “mix” the observed beam samples. Ideally we would like to see no “mixing” between pick-up and kicker and full “mixing” between kicker and pick-up. This again leads to a compromise for an optimum cooling time:

$$\frac{1}{\tau} = \frac{2W}{N}\left[2g - g^2(M+U)\right]$$

M is the mixing factor. It is the number of turns it takes a particle to pass from one sample to the next.

M = 1 (Complete mixing)

M>>1 (poor mixing)

If M >1, as will always be the case, we must again reduce the amplifier gain, λ, to continue cooling.

There is one more feature we should not forget. The particles have a predetermined velocity and so it takes a fixed time for the particle to travel from the pick-up to the kicker. The correction signal must be delayed to arrive at exactly the right moment in the kicker to correct the sample from which it was measured. Therefore very precise delays are needed. This is even more complicated if the system must cool at different momenta, as the delays will need to change as a function of beam momentum.

Longitudinal Cooling

Here all the same arguments apply, except that we use a longitudinal pick-up and a longitudinal kicker, which accelerates or decelerates the beam samples.
Longitudinally the beam has a frequency distribution, which mirrors it's momentum distribution. This frequency distribution we detect as the longitudinal schottky noise signal. This signal is then put through an amplifier with a filtered response as shown in figure 7.

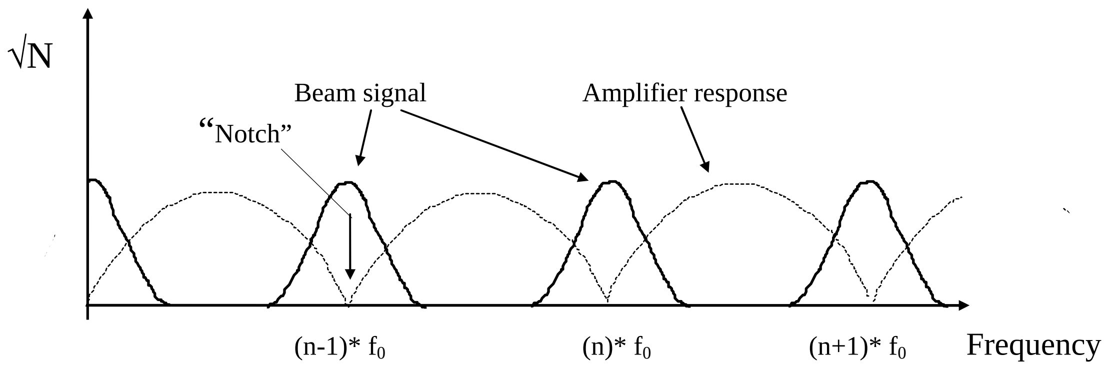


Figure 7: Longitudinal schottky noise and the filtered amplifier response

If the momentum, or frequency, of the sample falls outside of the central notch in the amplifier filter, then the kicker will accelerate or decelerate the sample accordingly. The gain of the amplifier, and hence amplitude of the correction, will be larger the further the sample frequency is from the centre of the notch. In this way, the system will tend to concentrate the beam at the notch frequency, which corresponds to the central or optimum beam momentum. Therefore it is very important that the notches fall exactly at the harmonics of the desired revolution frequency, otherwise the beam will be "cooled" to the wrong momentum.

Suggestions for further reading

As I have been very lax and not included any references in this text, for those who are interested in reading more on the topics covered here, I would recommend the following, in which you will find all of the ideas introduced here covered in far greater detail.

For a good general introduction:-

- Proton Synchrotron Accelerator theory. E.J.N. Wilson, CERN academic training program 1975-76 CERN 77-07
- CAS General Accelerator Physics course. Paris September 1984 CERN Yellow report 85-19 (two volumes).
- CAS Second General Accelerator Physics course. Aarhus September 1986 CERN Yellow report 87-10

For beam instabilities:-

- Theoretical aspects of the behaviour of beams in accelerators and storage rings. International school of Particle Accelerators ‘Ettore Majorana’ Centre for Scientific Culture, Erice November 1976
- CERN Yellow report 77-13

For stochastic cooling:-

- CAS Antiprotons for colliding beam facilities, Geneva, October 1983 CERN Yellow report 84-15.

For transfer lines:-

- Introduction to transfer lines and circular machines. P.J. Bryant CERN academic training program 1983-84 CERN Yellow report 84-04

For electron rings:-

- The Physics of Electron Storage Rings. M. Sands SLAC report 121, November 1970