# Electron Positron Scattering
# Electron Proton Scattering

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# What we do today

Calculate, using what we saw before, $e^+e^- \to \mu^+ \mu^-$
(and also, later, $e^-p \to e^-p$)

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

Fermi's Golden rule:
transitions between states

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \cdots.$$

$V_{ji}$ non-invariant matrix element
$\mathcal{M}_{ji}$ Lorentz invariant matrix element

$$V_{ji} = \mathcal{M}_{ji} \prod_k (2E_k)^{-1/2}$$

$e^-\tau^- \to e^-\tau^-$

Transition matrix

$$\mathcal{M} = -[Q_e e\, \overline{u}_e(p_3)\gamma^\mu u_e(p_1)] \frac{g_{\mu\nu}}{q^2} [Q_\tau e\, \overline{u}_\tau(p_4)\gamma^\nu u_\tau(p_2)].$$

Define currents

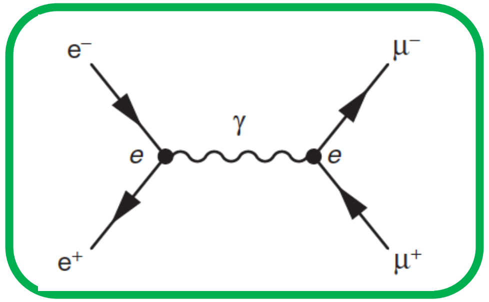
$$j_e^\mu = \overline{u}_e(p_3)\gamma^\mu u_e(p_1) \quad \text{and} \quad j_\tau^\nu = \overline{u}_\tau(p_4)\gamma^\nu u_\tau(p_2).$$

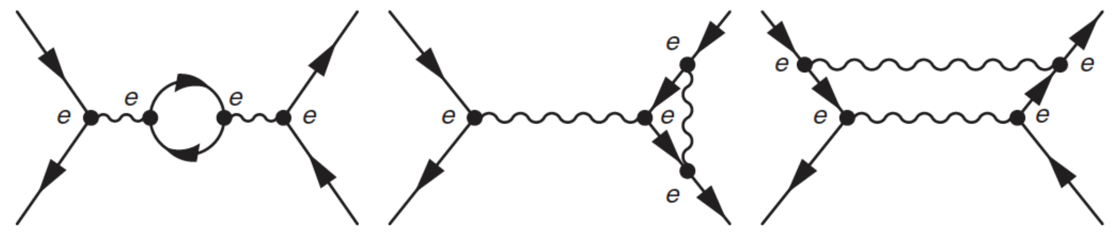
Rewrite in compact form $\mathcal{M} = -Q_e Q_\tau e^2 \dfrac{j_e \cdot j_\tau}{q^2}$

Rules:

$e^+ e^- \to \mu^+ \mu^-$

- Draw the Feynman diagram(s) representing the reaction. The one with the fewest vertices is the '*Lowest Order (LO)*' diagram; each vertex contributes a factor $e^2 \propto \alpha_{EM}$ to cross section and decay rates; the diagram below → $|\mathcal{M}|^2 \propto \alpha_{EM}{}^2$;
- higher order Feynman diagrams (Next To Leading, NLO) → 4 vertices $|\mathcal{M}|^2 \propto \alpha_{EM}{}^4$; see below;
- Compute Matrix Element squared using Feynman's rules.




General expression:

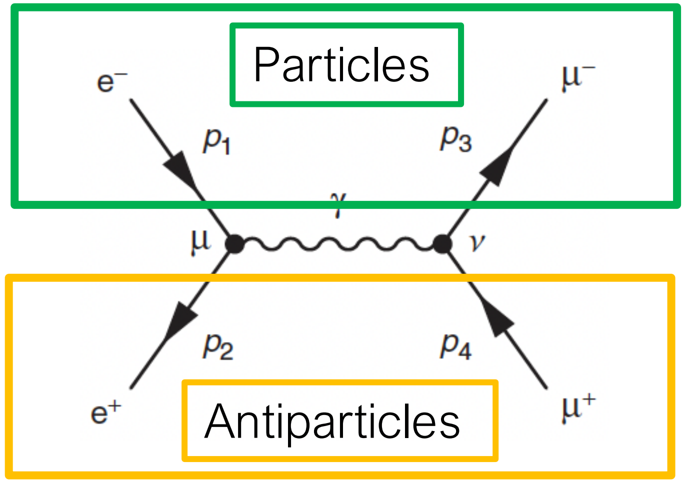
$$\mathcal{M}_{fi} = \mathcal{M}_{\mathrm{LO}} + \sum_j \mathcal{M}_{1,j} + \cdots$$

$$|\mathcal{M}_{fi}|^2 = \left(\alpha M_{\mathrm{LO}} + \alpha^2 \sum_j M_{1,\mathrm{j}} + \cdots\right)\left(\alpha M^*_{\mathrm{LO}} + \alpha^2 \sum_k M^*_{1,\mathrm{k}} + \cdots\right)$$

$$= \alpha^2 |M_{\mathrm{LO}}|^2 + \alpha^3 \sum_j \left(M_{\mathrm{LO}} M^*_{1,j} + M^*_{\mathrm{LO}} M_{1,j}\right) + \alpha^4 \sum_{jk} M_{1,j} M^*_{1,\mathrm{k}} + \cdots.$$

# Higher Order Diagrams & Interference

- QED α ≈ 1/137 → LO term dominates;
- The interference between the LO and the NLO diagrams is suppressed by a α ≈ 1/137;
- LO diagram accurate to ~ 1%.




- Particles $u$ spinors
- Antiparticles $v$ spinors

$$-i\mathcal{M} = [\bar{v}(p_2)\{ie\gamma^\mu\}u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3)\{ie\gamma^\nu\}v(p_4)]$$

Initial state    Final state

$$j_e^\mu = \bar{v}(p_2)\gamma^\mu u(p_1) \quad \text{and} \quad j_\mu^\nu = \bar{u}(p_3)\gamma^\nu v(p_4).$$

$$= -\frac{e^2}{q^2} g_{\mu\nu} j_e^\mu j_\mu^\nu,$$

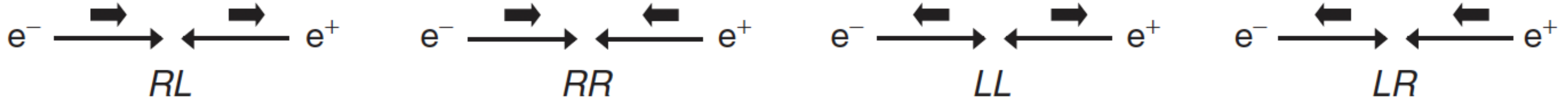
$$\mathcal{M} = -\frac{e^2}{s} j_e \cdot j_\mu.$$

**Spin**

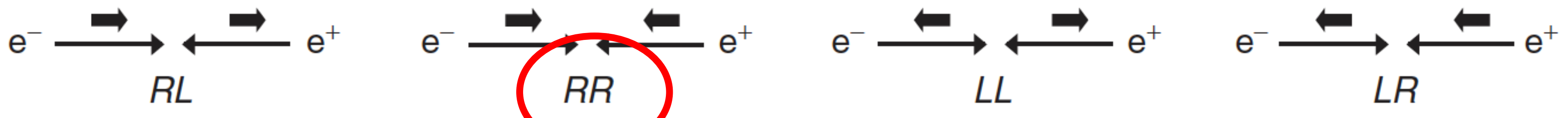
Consider possible combinations of each of the 2 initial / final state fermions
4x4=16 combinations

!

Initial state



# Spin combinations




Spin states are orthogonal → no interference → $|\mathcal{M}|^2$ for each combination considered independently;

For ONE particular spin configuration of the $e^+e^-$ state you have to sum $|\mathcal{M}|^2$ of the 4 possible $\mu^+\,\mu^-$ states.

Example: RR
$$\sum |\mathcal{M}_{RR}|^2 = |\mathcal{M}_{RR\to RR}|^2 + |\mathcal{M}_{RR\to RL}|^2 + |\mathcal{M}_{RR\to LR}|^2 + |\mathcal{M}_{RR\to LL}|^2$$

In modern colliders $e^+e^-$ are in general not polarised, equal amount of all possible states

→ average of terms

$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}\left(|\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LR}|^2 + |\mathcal{M}_{LL}|^2\right)$$
$$= \frac{1}{4}\left(|\mathcal{M}_{RR\to RR}|^2 + |\mathcal{M}_{RR\to RL}|^2 + \cdots + |\mathcal{M}_{RL\to RR}|^2 + \cdots\right)$$

$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2,$$

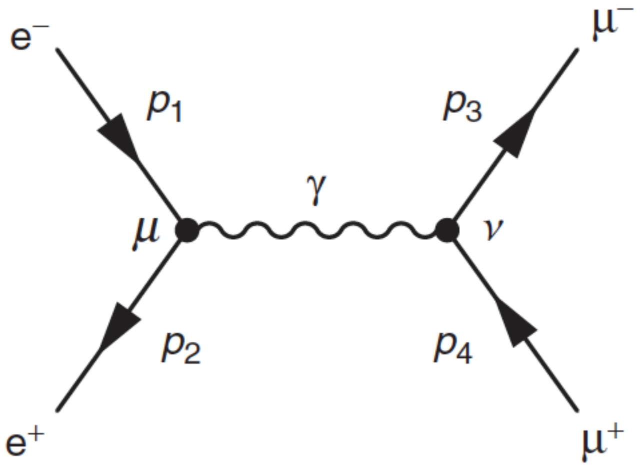
# Do Real Calculation!

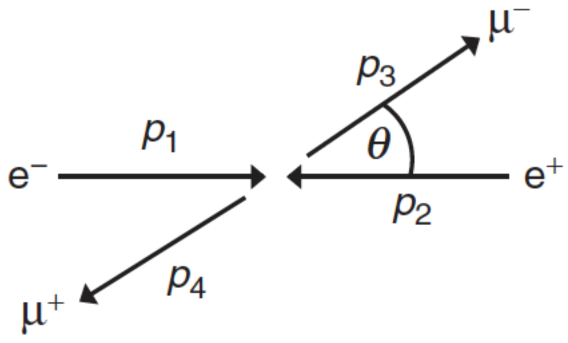
$s = \sin\left(\frac{\theta}{2}\right)$ and $c = \cos\left(\frac{\theta}{2}\right)$

In the ultra-relativistic region $(E \gg m) \rightarrow$ 
$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix},\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix},\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix},\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix},$$

These are the spinors we have to use in $j_e^\mu$ and $j_\mu^\nu$

$$j_{\mathrm{e}}^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \quad \text{and} \quad j_\mu^\nu = \overline{u}(p_3)\gamma^\nu v(p_4).$$




Initial state electron $e^-$: $\theta = 0, \varphi = 0$
initial-state positron $e^-$: $\theta = \pi, \varphi = \pi$

$$p_1 = (E, 0, 0, E),$$
$$p_2 = (E, 0, 0, -E),$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta),$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

Scattering observed in the Centre Of Mass system: $p_1 = -p_2$

# Spinors fermions & antifermions

General form for the spinors

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}, \quad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}, \quad v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}, \quad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix},$$

Initial state electron $e^-: \theta = 0, \varphi = 0$

initial-state positron $e^+: \theta = \pi, \varphi = \pi$

$s = \sin\frac{\theta}{2}$ and $c = \cos\frac{\theta}{2}$.

Possible spin states fermion

Possible spin states antifermion

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

Final state muon $\mu^-: \theta, \varphi = 0$

Final state anti-muon $\mu^+: \pi - \theta, \varphi = \pi$

Possible spin states fermion

Possible spin states antifermion

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}, \quad u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}, \quad v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}, \quad v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$$

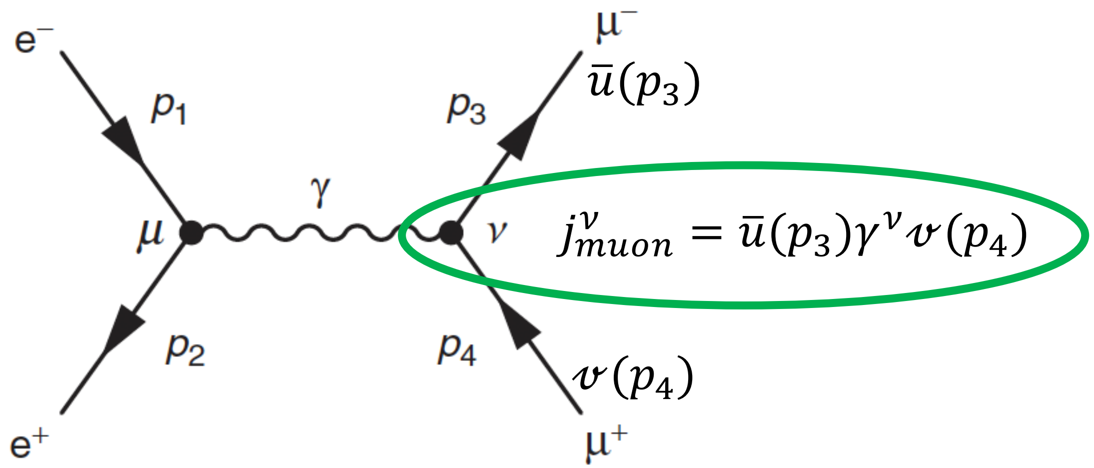
$$\sin\left(\frac{\pi - \theta}{2}\right) = \cos\left(\frac{\theta}{2}\right), \quad \cos\left(\frac{\pi - \theta}{2}\right) = \sin\left(\frac{\theta}{2}\right) \quad \text{and} \quad e^{i\pi} = -1,$$

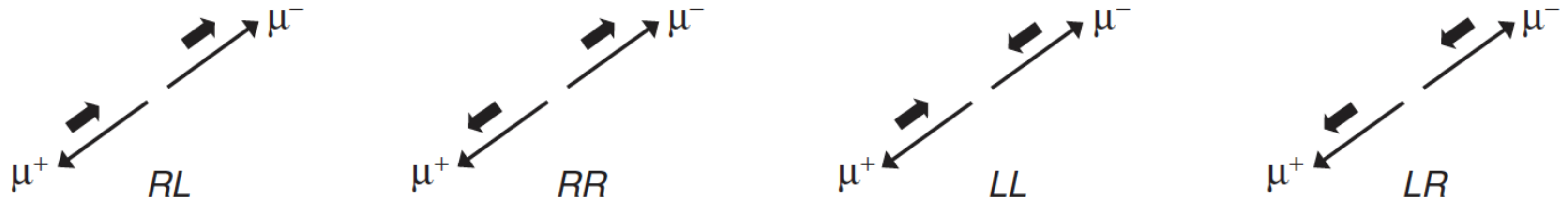
$$\mathcal{M} = -\frac{e^2}{s} j_e \cdot j_\mu$$

The muon current

$$j_\mu^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$$

Has to be computed for 4 possible spin configurations



# Calculating Muon Currents

$$\bar{\psi}\gamma^{\mu}\phi \equiv \psi^{\dagger}\gamma^{0}\gamma^{\mu}\phi$$

$j^{\nu}_{muon} = \bar{u}(p_3)\gamma^{\nu}v(p_4)$

Is a four-vector

$$u_{\uparrow}(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix},\ u_{\downarrow}(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix},\ v_{\uparrow}(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix},\ v_{\downarrow}(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},\quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$\psi$ first Dirac spinor
$\phi$ second Dirac spinor

$\psi^{\dagger}$ Hermitian conjugate
$\bar{\psi}$ Adjoint spinor $=\psi^{\dagger}\gamma_0$

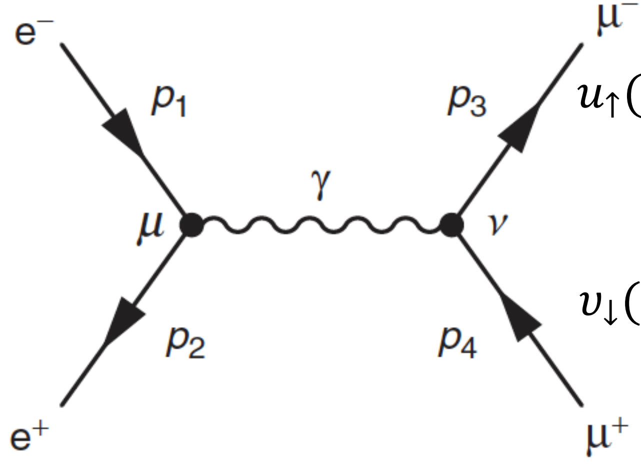
$\psi^{\dagger} = (\psi^{*})^{T}$

$$\bar{\psi}\gamma^0\phi = \psi^{\dagger}\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4,$$
$$\bar{\psi}\gamma^1\phi = \psi^{\dagger}\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1,$$
$$\bar{\psi}\gamma^2\phi = \psi^{\dagger}\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1),$$
$$\bar{\psi}\gamma^3\phi = \psi^{\dagger}\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2.$$

# Calculating Muon Currents *RL*

Example,for the RL combination:

- $\mu^-$ right-handed helicity state $\rightarrow u_\uparrow(p_3)$
- $\mu^+$ left-handed helicity state $\rightarrow v_\downarrow(p_4)$




$u_\uparrow(p_3)\,\sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}$

$v_\downarrow(p_4)\,\sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}.$

$$j^0_\mu = \bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0,$$
$$j^1_\mu = \bar{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E\cos\theta,$$
$$j^2_\mu = \bar{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE(-c^2 - s^2 - c^2 - s^2) = 2iE,$$
$$j^3_\mu = \bar{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta.$$

$$j_{\mu,RL} = \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta).$$

Similar calculation for 3 other combinations

$$j_{\mu,RL} = \bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta),$$
$$j_{\mu,RR} = \bar{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) = (0,0,0,0),$$
$$j_{\mu,LL} = \bar{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) = (0,0,0,0),$$
$$j_{\mu,LR} = \bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta).$$

Only two components of the current are different from 0!
RL and LR

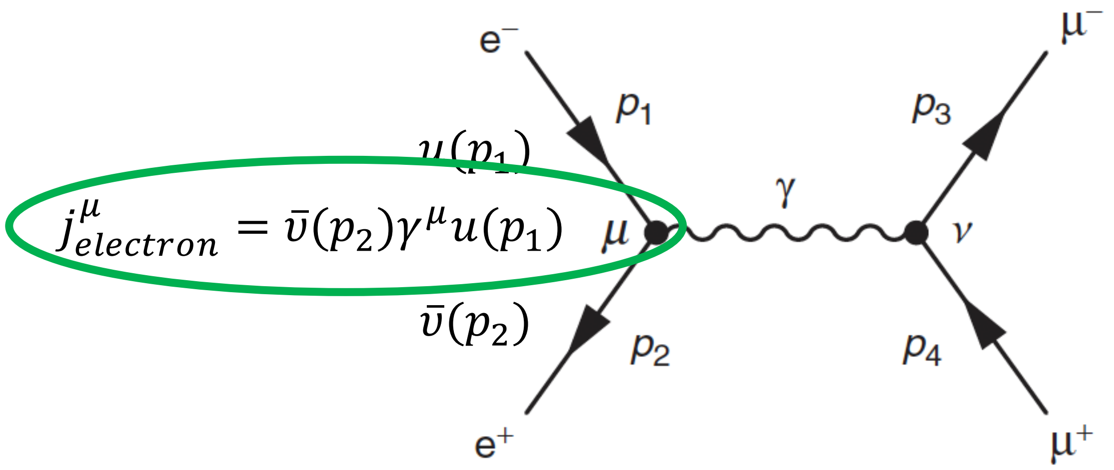
# Computing the Electron Current

$$\mathcal{M} = -\frac{e^2}{s} j_e \cdot j_\mu$$

Almost identical calculation can be repeated for the electron current:

$$j^{\mu}_{electron} = \bar{v}(p_2)\gamma^{\mu}u(p_1)$$

With a similar result to the muon case.




$$j_{e,RL} = \bar{v}_{\downarrow}(p_2)\gamma^{\mu}u_{\uparrow}(p_1) = 2E(0,-1,-i,0),$$
$$j_{e,LR} = \bar{v}_{\uparrow}(p_2)\gamma^{\mu}u_{\downarrow}(p_1) = 2E(0,-1,i,0).$$

Only RL and LR electron currents are non-zero

$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}\left(|\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LR}|^2 + |\mathcal{M}_{LL}|^2\right)$$
$$= \frac{1}{4}\left(|\mathcal{M}_{RR\to RR}|^2 + |\mathcal{M}_{RR\to RL}|^2 + \cdots + |\mathcal{M}_{RL\to RR}|^2 + \cdots\right)$$

# $\mathcal{M}_{fi}$ for RL→RL

$\mathcal{M}_{fi}$ superposition of 16 components

$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}\left(|\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LR}|^2 + |\mathcal{M}_{LL}|^2\right)$$
$$= \frac{1}{4}\left(|\mathcal{M}_{RR\to RR}|^2 + |\mathcal{M}_{RR\to RL}|^2 + \cdots + |\mathcal{M}_{RL\to RR}|^2 + \cdots\right)$$

One component: RL→RL $\quad \mathcal{M}_{RL\to RL}$ for the process $e^-_\uparrow e^+_\downarrow \to \mu^-_\uparrow \mu^+_\downarrow$

Compute scalar product of Electron & muon currents

$$j^{\mu}_{e,RL} = \overline{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) = 2E(0,-1,-i,0),$$
$$j^{\nu}_{\mu,RL} = \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta, i, \sin\theta).$$

(Center Of Mass energy $s = 4E^2$)

$$\mathcal{M} = -\frac{e^2}{s} j_e \cdot j_\mu$$

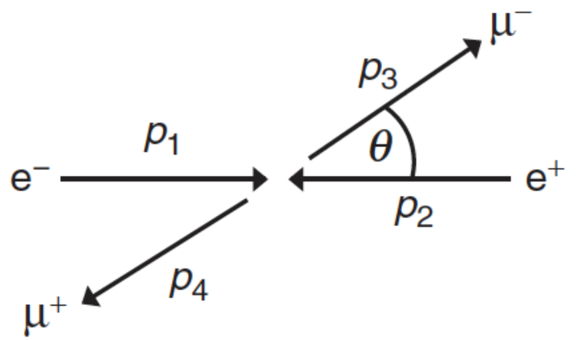
$$\mathcal{M}_{RL\to RL} = -\frac{e^2}{s}[2E(0,-1,-i,0)]\cdot[2E(0,-\cos\theta,i,\sin\theta)]$$
$$= e^2(1+\cos\theta)$$
$$= 4\pi\alpha(1+\cos\theta).$$

# $\mathcal{M}_{fi}$

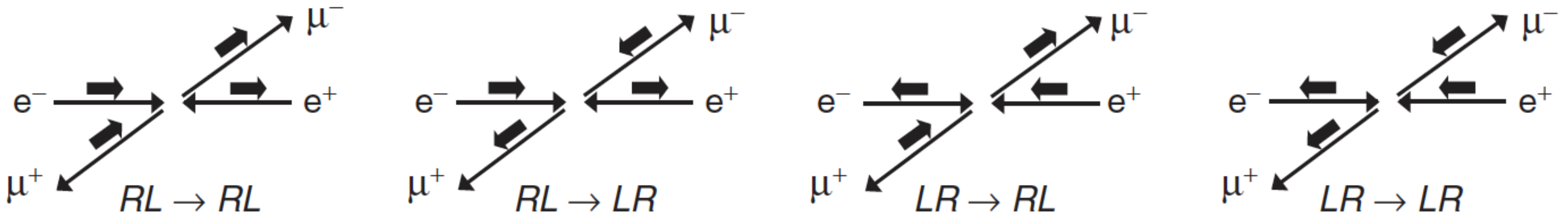
Similarly

$$|\mathcal{M}_{RL\to RL}|^2 = |\mathcal{M}_{LR\to LR}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2,$$
$$|\mathcal{M}_{RL\to LR}|^2 = |\mathcal{M}_{LR\to RL}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2,$$

$\theta$ is the angle of the outgoing $\mu^-$ with respect to the direction of the incoming $e^-$




The only non-zero components of $\mathcal{M}_{fi}$ are




The spin-averaged matrix element $\mathcal{M}_{fi}$ is

$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}\times\left(|\mathcal{M}_{RL\to RL}|^2 + |\mathcal{M}_{RL\to LR}|^2 + |\mathcal{M}_{LR\to RL}|^2 + |\mathcal{M}^2_{LR\to LR}|\right)$$
$$= \frac{1}{4}e^4\left[2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right]$$
$$= e^4(1+\cos^2\theta).$$

# The cross section of the process $e^+e^- \to \mu^+ \mu^-$

We found already this general expression for process

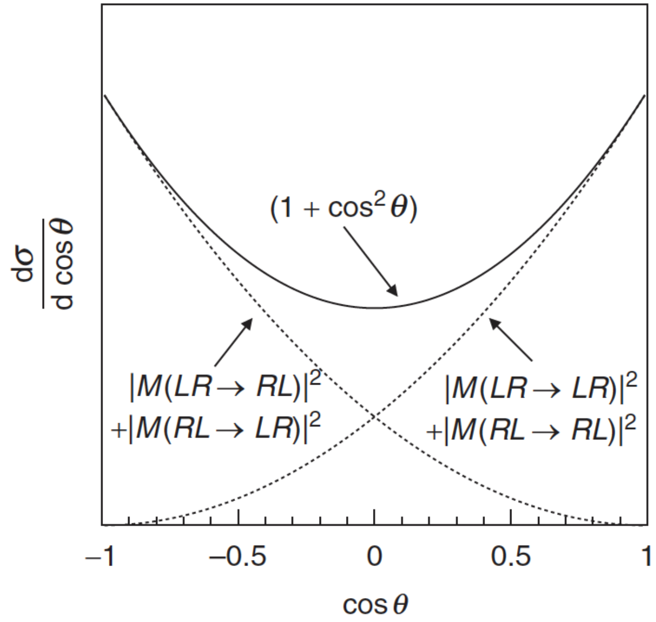
$$a + b \to c + d$$

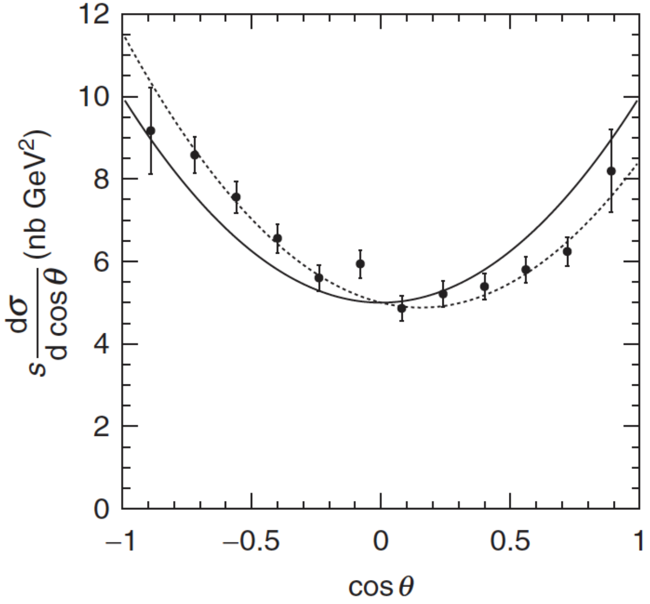
$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |\mathcal{M}_{fi}|^2,$$

- $p_f^*$ momentum of the final state particle in the cms system
- $p_i^*$ momentum of the initial state particle in the cms system

If we use $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} e^4 (1 + \cos^2\theta)$, and $p_f^* = p_i^* = E$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} e^4 (1 + \cos^2\theta), \quad \Rightarrow \quad \alpha = e^2/(4\pi), \quad \Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\theta).$$




The measured $e^- \to \mu^+ \mu^-$ differential cross section at √s = 34.4 GeV from the JADE experiment, adapted from Bartel et al. (1985). The solid curve is the lowest-order QED prediction. The dotted curve includes electroweak corrections.

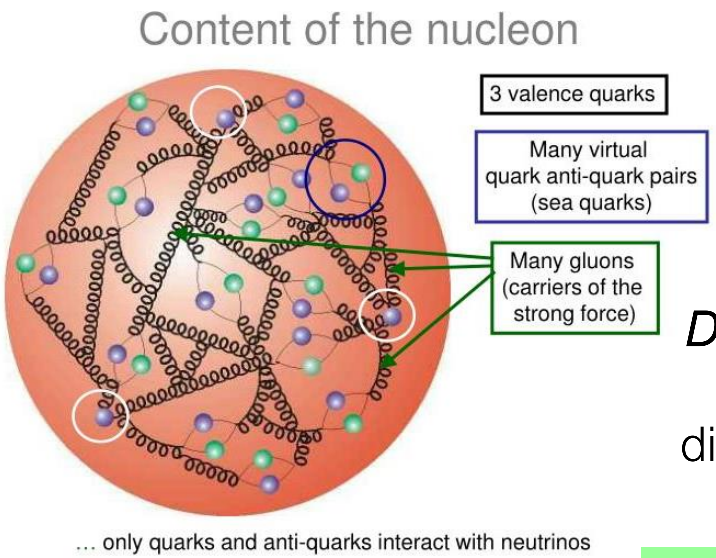
# Electron – Proton scattering

The proton has a very complex structure.
How to study it?

- How deep the virtual particle can penetrate the proton depends on the equivalent wavelength of the exchanged virtual photon:

$$\lambda \ll R \;\rightarrow\; Q^2 \gg \hbar^2/R^2$$

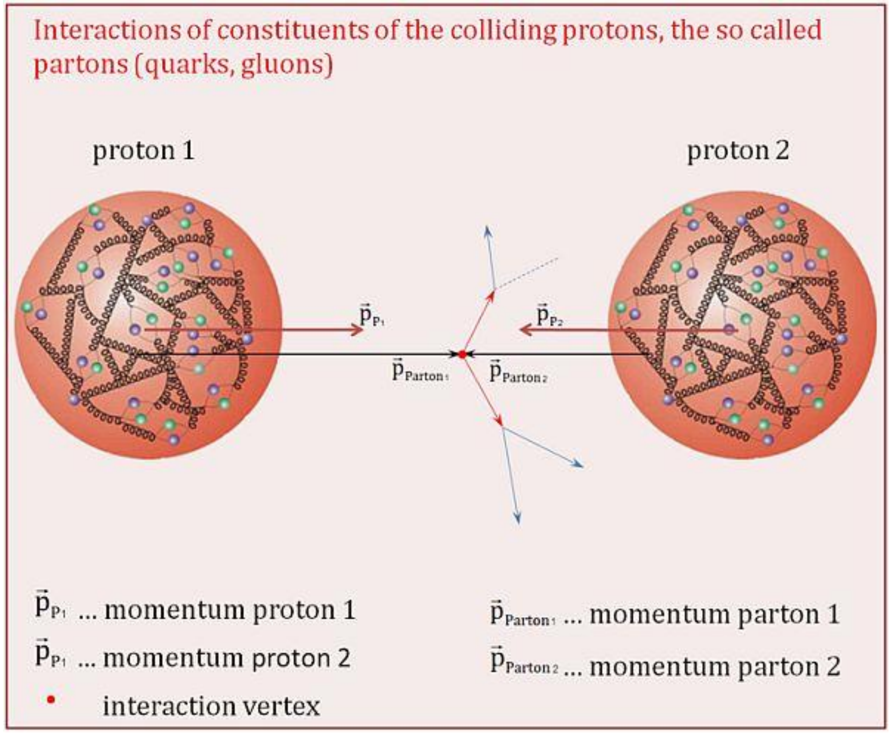
- At low energies, ***elastic scattering***, the proton remains intact. Interaction between a photon and a proton (as a whole) → global properties of the proton such as radius.
- At high energies, ***deep inelastic scattering***, the proton breaks up. Is interpreted as the elastic scattering of the electron from one of the quarks within the proton.




***Deep inelastic scattering*** gives the momentum distribution of the quarks.

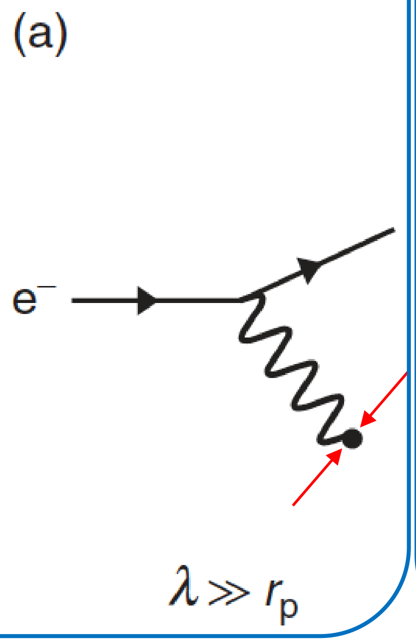
$e^- p$ best way to study proton

Proton – proton scattering

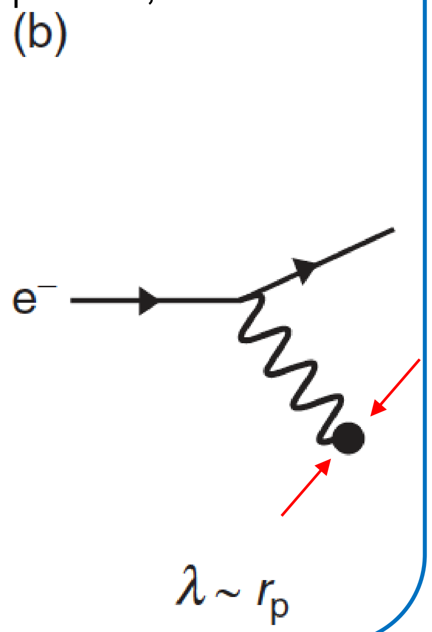


# $e^- p$ Scattering

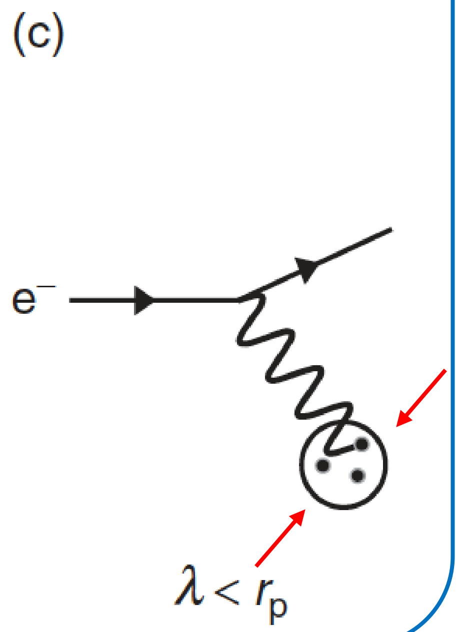
$\lambda \gg r_p$
$\rightarrow eP \rightarrow eP$ elastic scattering on a point-like proton;



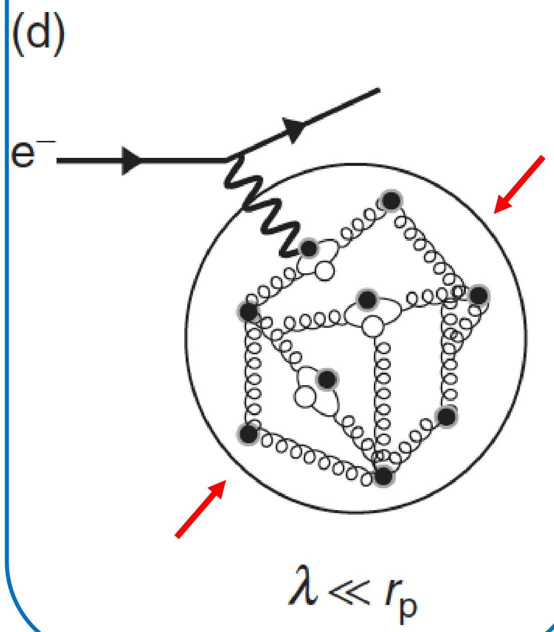

$\lambda \approx r_p$
$\rightarrow eP \rightarrow eP$ no longer purely electrostatic account for the extended charge and magnetic moment of the proton;




$\lambda < r_p$
$\rightarrow eP \rightarrow eP$ elastic + (small) interaction with proton constituents;



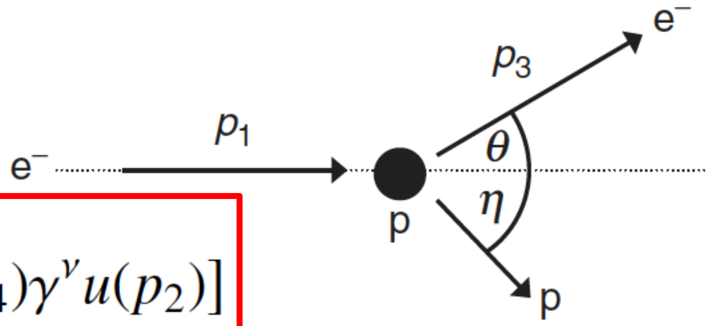

$\lambda \ll r_p$
$\rightarrow eP \rightarrow eP$ detailed structure of the proton; elastic scattering electron-quark

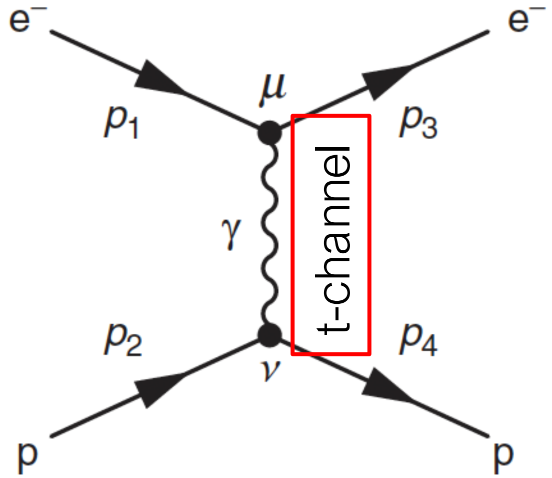


# Rutherford & Mott Cross Sections

Exactly the same technique as for $e^+e^- \rightarrow \mu^+ \mu^-$

Rutherford & Mott non-relativistic limit
assume proton ~ point-like fermion




$$\mathcal{M}_{fi} = \frac{Q_q e^2}{q^2} [\overline{u}(p_3)\gamma^\mu u(p_1)]\, g_{\mu\nu}\, [\overline{u}(p_4)\gamma^\nu u(p_2)]$$

Spinors of the incoming electron exactly those we used in $e^+e^- \rightarrow \mu^+ \mu^-$

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix} \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \end{pmatrix},$$

$$u_\uparrow = N_e \begin{pmatrix} c \\ se^{i\phi} \\ \kappa c \\ \kappa s e^{i\phi} \end{pmatrix} \qquad u_\downarrow = N_e \begin{pmatrix} -s \\ ce^{i\phi} \\ \kappa s \\ -\kappa c e^{i\phi} \end{pmatrix}$$

$s = \sin(\theta/2)$ and $c = \cos(\theta/2)$

$N_e = \sqrt{E + m_e}$,

Introduced factor $K$: $\kappa = \dfrac{p}{E + m_e} \equiv \dfrac{\beta_e \gamma_e}{\gamma_e + 1}$

Factor $K$ distinguishes between

- Very low energy $\rightarrow K \approx 0.$;
- Ultra high-energy $\rightarrow K \approx 1.$;

# Reminder (few slides before)

Calculate currents

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$\psi$ first spinor
$\phi$ second spinor

$$\overline{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4,$$

$$\overline{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1,$$

$$\overline{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1),$$

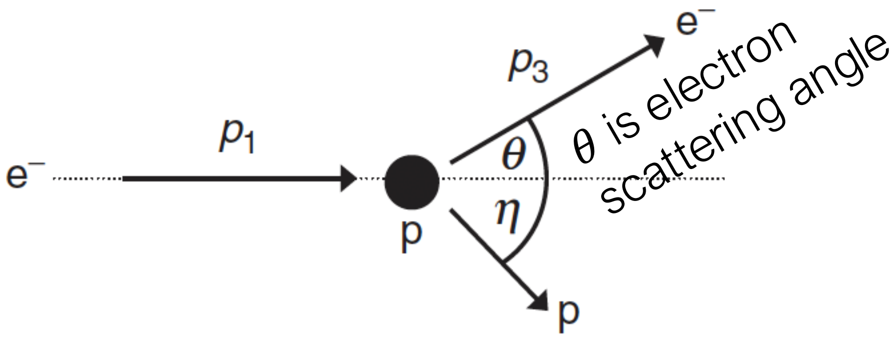
$$\overline{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2.$$

# Rutherford & Mott Scattering

$s = \sin(\theta/2)$ and $c = \cos(\theta/2)$

$c_\eta = \cos(\eta/2)$ and $s_\eta = \sin(\eta/2)$

If consider the scattering to be elastic & proton at rest, then the energy of the electron doesn't change.




$$u_\uparrow(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ \kappa \\ 0 \end{pmatrix}, \quad u_\downarrow(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\kappa \end{pmatrix} \quad \text{and} \quad u_\uparrow(p_3) = N_e \begin{pmatrix} c \\ s \\ \kappa c \\ \kappa s \end{pmatrix}, \quad u_\downarrow(p_3) = N_e \begin{pmatrix} -s \\ c \\ \kappa s \\ -\kappa c \end{pmatrix}.$$

All possible electron combinations are:

$$j_{e\uparrow\uparrow} = \bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E + m_e)\left[(\kappa^2 + 1)c, 2\kappa s, +2i\kappa s, 2\kappa c\right],$$

$$j_{e\downarrow\downarrow} = \bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E + m_e)\left[(\kappa^2 + 1)c, 2\kappa s, -2i\kappa s, 2\kappa c\right],$$

$$j_{e\downarrow\uparrow} = \bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E + m_e)\left[(1 - \kappa^2)s, 0, 0, 0\right],$$

$$j_{e\uparrow\downarrow} = \bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E + m_e)\left[(\kappa^2 - 1)s, 0, 0, 0\right].$$

In the relativistic limit $K \approx 1 \rightarrow$ $j_{e\downarrow\uparrow}$ and $j_{e\uparrow\downarrow}$ are both 0

At low electron energy all 4 currents contribute

Proton spinors are = electron spinors ($\sqrt{E_p + m_p} \approx \sqrt{2m_p}$) $\quad K_p \approx 0 \rightarrow$ lower 2 components of the spinors are 0

$$u_\uparrow(p_2) = \sqrt{2m_p}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv u_1(p_2) \quad \text{and} \quad u_\downarrow(p_2) = \sqrt{2m_p}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \equiv u_2(p_2).$$

$$u_\uparrow(p_4) \approx \sqrt{2m_p}\begin{pmatrix} c_\eta \\ -s_\eta \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_\downarrow(p_4) \approx \sqrt{2m_p}\begin{pmatrix} -s_\eta \\ -c_\eta \\ 0 \\ 0 \end{pmatrix}$$

$$\mathcal{M}_{fi} = \frac{e^2}{q^2} j_e \cdot j_p$$

The 4 possible combinations of the proton current are

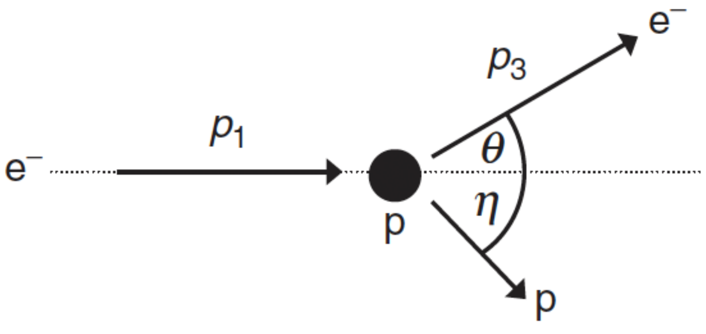
$$j_{p\uparrow\uparrow} = -j_{p\downarrow\downarrow} = 2m_p\left[c_\eta, 0, 0, 0\right] \quad \text{and} \quad j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = -2m_p\left[s_\eta, 0, 0, 0\right]$$

Spin-averaged $\mathcal{M}_{fi}$

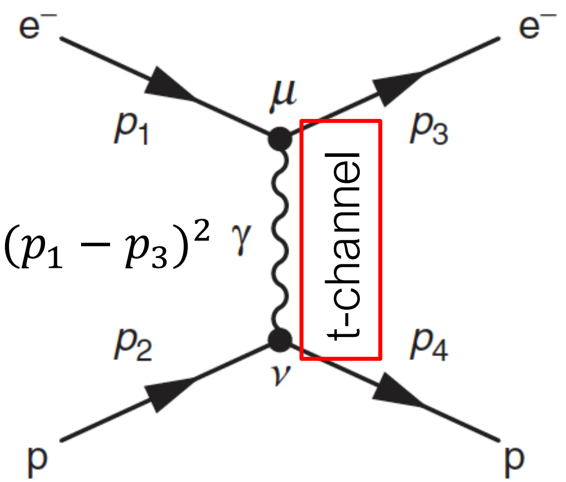
$$\begin{aligned}\langle|\mathcal{M}_{fi}^2|\rangle &= \frac{1}{4}\sum|\mathcal{M}_{fi}^2| \\ &= \frac{1}{4}\frac{e^4}{q^4} \times 4m_p^2(E+m_e)^2 \cdot \left[c_\eta^2 + s_\eta^2\right] \cdot \left[4(1+\kappa^2)^2 c^2 + 4(1-\kappa^2)^2 s^2\right] \\ &= \frac{4m_p^2 m_e^2 e^4(\gamma_e+1)^2}{q^4}\left[(1-\kappa^2)^2 + 4\kappa^2 c^2\right],\end{aligned}$$

Can be simplified using/defining $\kappa = \frac{\beta_e\gamma_e}{\gamma_e+1}$ and $(1-\beta_e^2)\gamma_e^2 = 1$, $E = \gamma_e m_e$. A bit of calculations!

$$\langle|\mathcal{M}_{fi}^2|\rangle = \frac{16m_p^2 m_e^2 e^4}{q^4}\left[1 + \beta_e^2\gamma_e^2\cos^2\frac{\theta}{2}\right]$$




$$q^2 = (p_1 - p_3)^2$$




For the elastic scattering process where the proton doesn't move,

$$E_1 = E_3 = E.$$
$$p_1 = p_3 = p.$$

$$q^2 = (0, \mathbf{p}_1 - \mathbf{p}_3)^2 = -2\mathrm{p}^2(1 - \cos\theta) = -4\mathrm{p}^2 \sin^2(\theta/2)$$

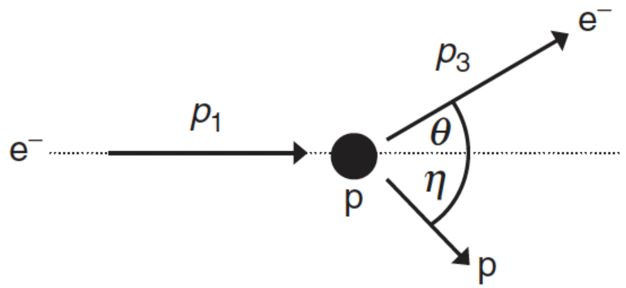
$$\langle|\mathcal{M}_{fi}^2|\rangle = \frac{m_\mathrm{p}^2 m_\mathrm{e}^2 e^4}{\mathrm{p}^4 \sin^4(\theta/2)}\left[1 + \beta_\mathrm{e}^2\gamma_\mathrm{e}^2 \cos^2\frac{\theta}{2}\right]$$

This expression is valid both
- When the electron is non-relativistic
- When the electron is relativistic

# Rutherford Scattering

The expression we just computed can represent two different cases:

- Rutherford scattering, the energy of the electron is very low, non-relativistic case;
- Mott scattering, the energy of the incoming/outgoing electron cannot be neglected.




Rutherford Scattering

$$\langle|\mathcal{M}_{fi}^2|\rangle = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4(\theta/2)}\left[1 - \beta_e^2\gamma_e^2\cos^2\frac{\theta}{2}\right] \quad \beta \approx 0. \rightarrow \quad \rightarrow \langle|\mathcal{M}_{fi}^2|\rangle = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4(\theta/2)}$$

The formula for the laboratory scattering (discussed in part#1)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{m_p + E_1 - E_1\cos\theta}\right)^2 |\mathcal{M}_{fi}|^2.$$

$$E_1 \approx m_e \ll m_p$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_p^2}\langle|\mathcal{M}_{fi}|^2\rangle = \frac{m_e^2 e^4}{64\pi^2 p^4 \sin^4(\theta/2)}$$

$$E_K = p^2/2m_e$$

Kinetic energy

$$e^2 = 4\pi\alpha$$

The same expression could have been derived from 'classical' scattering

electron $V(r) = \alpha/r$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16E_K^2 \sin^4(\theta/2)}.$$

→ no spin!

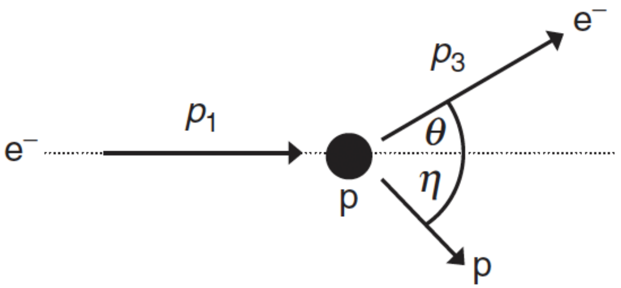
Angular dependence originates from the $1/q^2$ propagator term.

# Mott Scattering

$$\kappa = \frac{\beta_e \gamma_e}{\gamma_e + 1} \quad \text{and} \quad (1 - \beta_e^2)\gamma_e^2 = 1$$

Mott scattering:
- Electron is relativistic;
- proton recoil can still be neglected.

$$\langle |\mathcal{M}_{fi}^2| \rangle = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4(\theta/2)} \left[ 1 + \beta_e^2 \gamma_e^2 \cos^2 \frac{\theta}{2} \right]$$




$\rightarrow m_e \ll E \ll m_p \; \rightarrow K \approx 1$

$u_\uparrow(p_1)$

$$u_\uparrow(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ \kappa \\ 0 \end{pmatrix}, \; u_\downarrow(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\kappa \end{pmatrix} \text{ and } u_\uparrow(p_3) = N_e \begin{pmatrix} c \\ s \\ \kappa c \\ \kappa s \end{pmatrix}, \; u_\downarrow(p_3) = N_e \begin{pmatrix} -s \\ c \\ \kappa s \\ -\kappa c \end{pmatrix}$$

$$j_{e\uparrow\uparrow} = \overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E + m_e)\left[(\kappa^2 + 1)c, 2\kappa s, +2i\kappa s, 2\kappa c\right],$$

$$j_{e\downarrow\downarrow} = \overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E + m_e)\left[(\kappa^2 + 1)c, 2\kappa s, -2i\kappa s, 2\kappa c\right],$$

$$j_{e\downarrow\uparrow} = \overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E + m_e)\left[(1 - \kappa^2)s, 0, 0, 0\right],$$

$$j_{e\uparrow\downarrow} = \overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E + m_e)\left[(\kappa^2 - 1)s, 0, 0, 0\right].$$

$$\kappa = \frac{p}{E + m_e} \equiv \frac{\beta_e \gamma_e}{\gamma_e + 1}$$

Reminder!

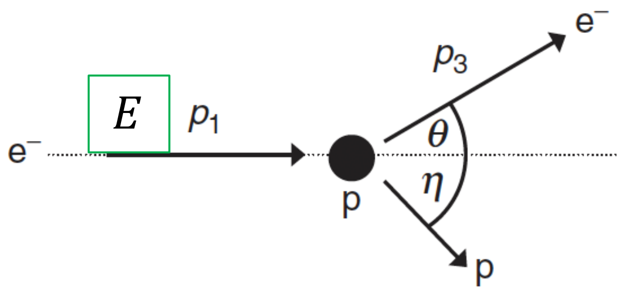
Same expressions for proton

→ (small) contribution from $\text{spin}_e$-$\text{spin}_p$ interaction

$$\langle|\mathcal{M}_{fi}^2|\rangle = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4(\theta/2)}\left[1 + \beta_e^2\gamma_e^2 \cos^2\frac{\theta}{2}\right]$$

$$\beta_e^2\gamma_e^2 \gg 1$$
$$E = \gamma_e m_e \; (\gamma_e = E/m_e)$$
$$E \approx p$$

$$\langle|\mathcal{M}_{fi}^2|\rangle \approx \frac{m_p^2 e^4}{E^2 \sin^4(\theta/2)} \cos^2\frac{\theta}{2}$$




$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{m_p + E_1 - E_1\cos\theta}\right)^2 |\mathcal{M}_{fi}|^2,$$

$$m_e \ll E \ll m_p$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4(\theta/2)}\cos^2\frac{\theta}{2}.$$

As for the Rutherford scattering this expression could have been derived from ‘classical’ scattering

electron $V(\boldsymbol{r}) = \alpha/\boldsymbol{r}$

→ almost no spin contribution!

# ep Scattering: Form Factors

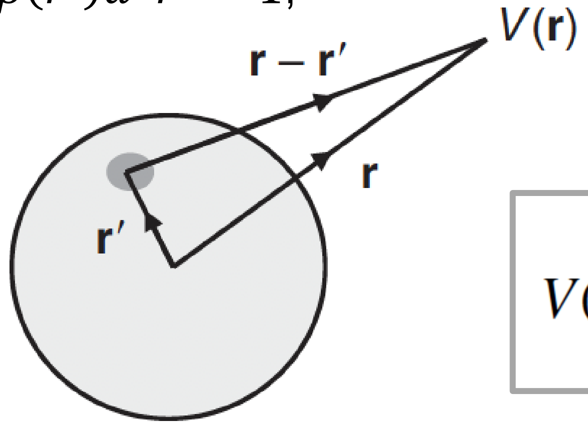
**Proton Structure → Form Factors**

Classical calculation of Rutherford and Mott scattering:

electron in the Coulomb $V(\boldsymbol{r}) = \alpha/\boldsymbol{r}$ point-like proton

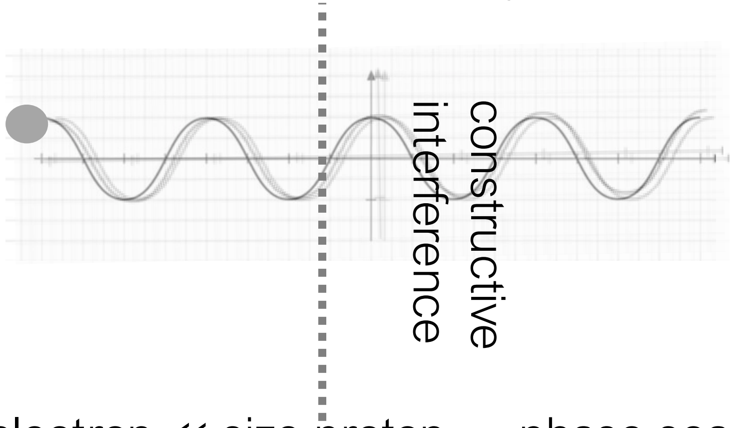
Classical treatment using static potential $V(\boldsymbol{r}) = \alpha/\boldsymbol{r}$

$Q$ total charge of 'extended' proton;
$r'$ distance from center of charge distribution;
$\rho(r')$ charge distribution;
$\int \rho(r') d^3r' = 1$;

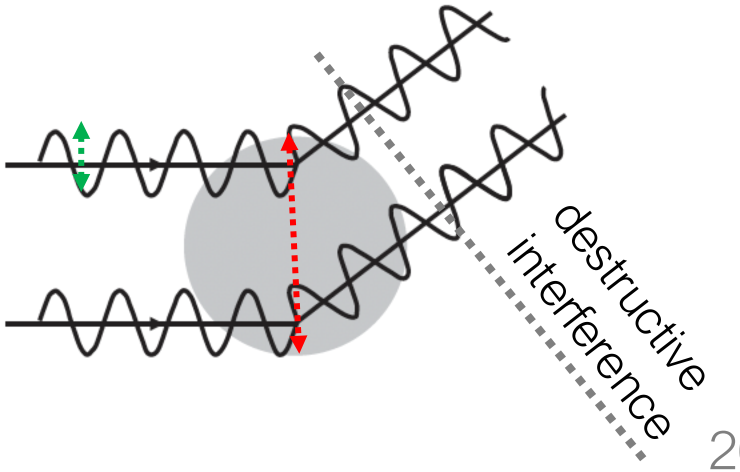



$$V(\mathbf{r}) = \int \frac{Q\rho(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' .$$

Wavelength of electron ≫ size proton → scattered wave in phase → add constructively;




Wavelength of electron ≪ size proton → phase scattered wave depends on position → add destructively → the negative interference greatly reduces the total amplitude.



# Form Factors - 2

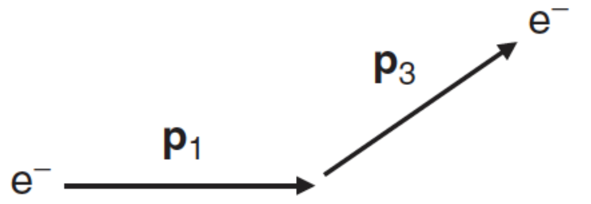
Initial & final electron ~classic wave

$$\psi_i = e^{i(p_1 \cdot r - Et)}$$
$$\psi_f = e^{i(p_3 \cdot r - Et)}$$

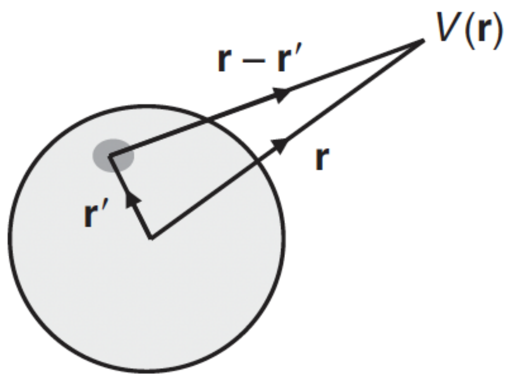
$$\boldsymbol{q} = \boldsymbol{p_1} - \boldsymbol{p_3}$$

$$-Et + Et$$

$$\mathcal{M}_{fi} = \langle \psi_f | V(\mathbf{r}) | \psi_i \rangle = \int \underbrace{e^{-i\mathbf{p}_3 \cdot \mathbf{r}}}_{\psi_f^*} V(\mathbf{r}) \underbrace{e^{i\mathbf{p}_1 \cdot \mathbf{r}}}_{\psi_i} \, \mathrm{d}^3\mathbf{r}$$




$$V(\mathbf{r}) = \int \frac{Q\rho(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3\mathbf{r}' \, .$$




$$\mathcal{M}_{fi} = \iint e^{i\mathbf{q}\cdot\mathbf{r}} \frac{Q\rho(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3\mathbf{r}' \mathrm{d}^3\mathbf{r}$$
$$= \iint e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{i\mathbf{q}\cdot\mathbf{r}'} \frac{Q\rho(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3\mathbf{r}' \mathrm{d}^3\mathbf{r} \, .$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$$

$$\mathcal{M}_{fi} = \int e^{i\mathbf{q}\cdot\mathbf{R}} \frac{Q}{4\pi|\mathbf{R}|} \mathrm{d}^3\mathbf{R} \int \rho(\mathbf{r}') e^{i\mathbf{q}\cdot\mathbf{r}'} \, \mathrm{d}^3\mathbf{r}' \, .$$

scattering from a potential due to a point charge.

$$F(\mathbf{q}^2) = \int \rho(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} \, \mathrm{d}^3\mathbf{r}.$$

Form Factor:
distribution of charge inside the proton

The form factor $F(\boldsymbol{q}^2)$ is the Fourier transform of the charge distribution.

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^{\mathrm{pt}} F(\mathbf{q}^2).$$

*Mott*: To account for the charge distribution inside the proton → Form Factor

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2\frac{\theta}{2}.$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \to \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} \cos^2\left(\frac{\theta}{2}\right) \left|F(\mathbf{q}^2)\right|^2.$$

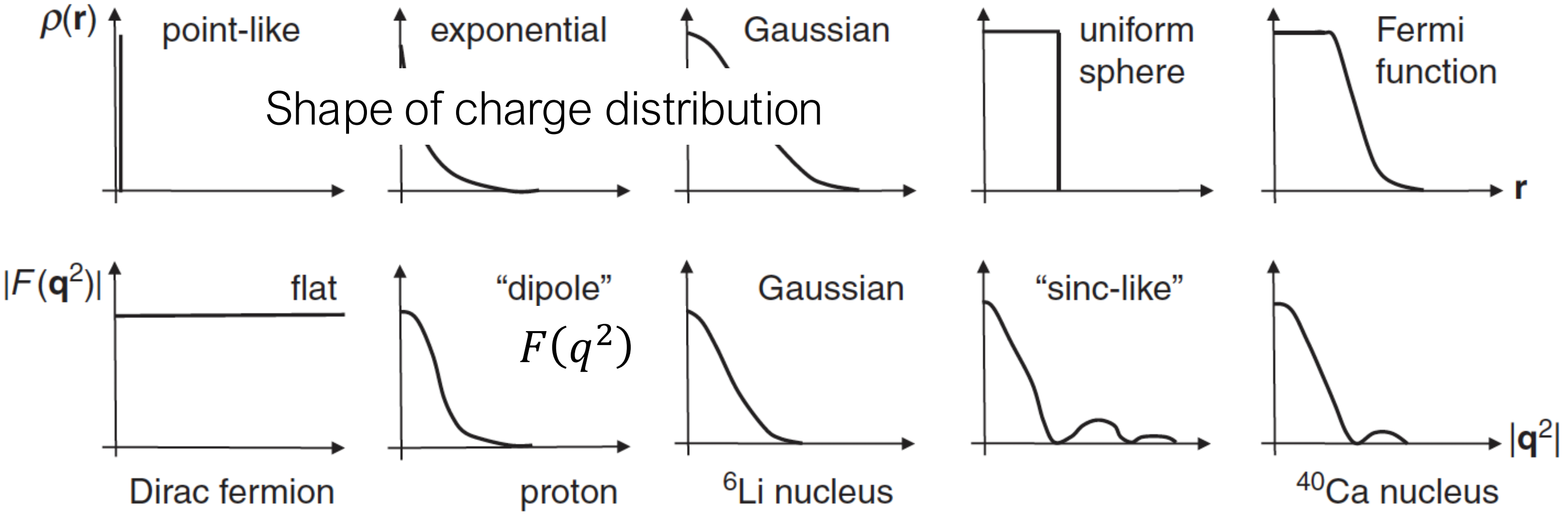
$$F(\mathbf{q}^2) = \int \rho(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}}\, d^3\mathbf{r}.$$

- If the electron wavelength is much **larger** than the size of the proton → $q \cdot r \approx 0.$ → point like source → $F(q^2) = 1;$
- If the electron wavelength is much **smaller** than the size of the proton → rapid change of scattered phases $F(q^2 \to \infty) = 0$ → reduction of amplitude

Measure

$$F(q^2) = \frac{(d\sigma/d\Omega)}{angular\ terms(\theta)}$$

→ derive shape of charge distribution by comparison

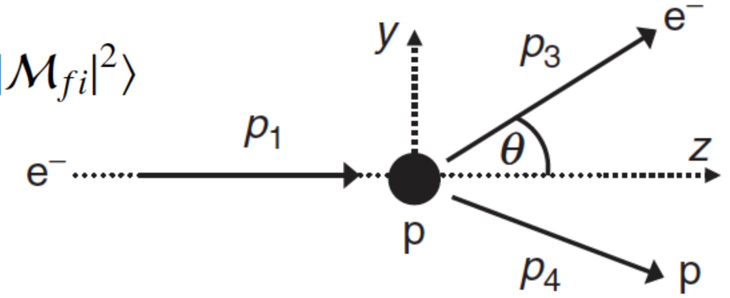


# Mott Scattering in the Relativistic case

*ep* **elastic** scattering at high energy:

- the recoil of the proton cannot be neglected and
- the magnetic spin–spin interaction becomes important.

$$\frac{d\sigma}{d\Omega} \approx \frac{1}{64\pi^2}\left(\frac{E_3}{m_p E_1}\right)^2 \langle|\mathcal{M}_{fi}|^2\rangle$$




$$p_1 = (E_1, 0, 0, E_1),$$
$$p_2 = (m_p, 0, 0, 0),$$
$$p_3 = (E_3, 0, E_3\sin\theta, E_3\cos\theta),$$
$$p_4 = (E_4, \mathbf{p}_4).$$

$$Q^2 \equiv -q^2 = 4E_1E_3\sin^2\frac{\theta}{2}$$

$$q^2 = -4E_1E_3\sin^2\frac{\theta}{2}.$$

Some calculation → book!

$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{m_p^2 e^4}{E_1E_3\sin^4(\theta/2)}\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right]$$

Mott formula in the non-relativistic case

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4(\theta/2)}\cos^2\frac{\theta}{2}.$$

Mott formula in the relativistic case

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4(\theta/2)}\frac{E_3}{E_1}\left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right)$$

Point-like proton, no $F(q^2)$

Two new terms appear:

- $\frac{E_3}{E_1}$ ($\frac{E_3}{E_1} \approx 1$ at low energy)
- and $Q^2/2m_p^2 \cdot \sin(\frac{\theta}{2})^2$ this is due to spin-spin interaction

# Rosenbluth Formula

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \frac{E_3}{E_1}\left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right)$$

Elastic $ep \to ep$ scattering, point-like spin-half proton.
*Contains terms due to the spin-spin interaction*;

Account also for the finite size of the proton → two form factors:

- one related to the charge distribution of the proton, $G_E(Q^2)$;
- and the other related to the magnetic moment distribution within the proton, $G_M(Q^2)$

$$\tau = \frac{Q^2}{4m_p^2}.$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)} \frac{E_3}{E_1}\left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right),$$

Define $(\frac{d\sigma}{d\Omega})_0$ Mott formula in the non-relativistic case

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)}\cos^2\frac{\theta}{2}.$$

Rewrite Rosenbluth formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)\cdot\left(\frac{d\sigma}{d\Omega}\right)_0$$

# Measuring $G_E(Q^2)$ and $G_M(Q^2)$

At low $Q^2 \rightarrow \tau = \frac{Q^2}{4m_p^2} \approx 0.$

$$\frac{d\sigma}{d\Omega} = \left(\frac{G_E^2 + \cancel{\tau} G_M^2}{(1+\cancel{\tau})} + \cancel{2\tau G_M^2 \tan^2\frac{\theta}{2}}\right)\cdot\left(\frac{d\sigma}{d\Omega}\right)_0$$

$$\frac{d\sigma}{d\Omega}\bigg/\left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2.$$

At high $Q^2 \rightarrow \tau = \frac{Q^2}{4m_p^2}$ is very large $\rightarrow$

$$\frac{d\sigma}{d\Omega}\bigg/\left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau\tan^2\frac{\theta}{2}\right)G_M^2$$

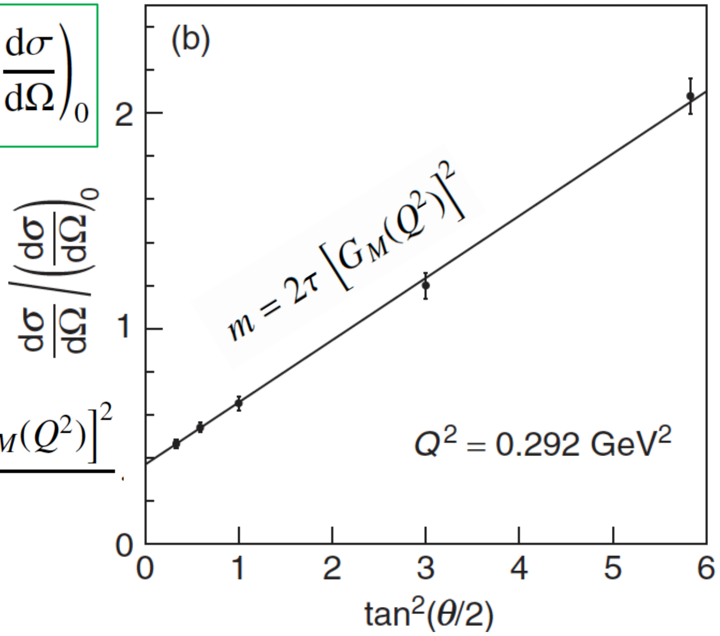
$E_1^2$ incoming electron energy
$\theta$ outgoing electron angle

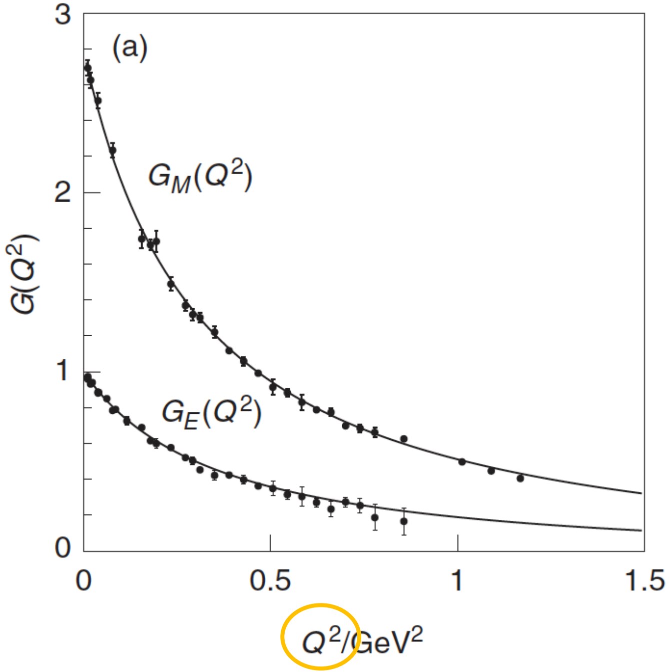
$$Q^2 = \frac{2m_p E_1^2(1-\cos\theta)}{m_p + E_1(1-\cos\theta)}.$$

By varying the initial electron energy and the electron scattering angle that corresponds to a given $Q^2 \rightarrow G_E(Q^2)$ and $G_M(Q^2)$

$$\frac{d\sigma}{d\Omega} = \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)\cdot\left(\frac{d\sigma}{d\Omega}\right)_0$$

$$c = \frac{\left[G_E(Q^2)\right]^2 + \tau\left[G_M(Q^2)\right]^2}{(1+\tau)}.$$



# Measuring $G_E$ & $G_M$