Interaction by Particle Exchange

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In this lecture

- How particles interact via exchange of particles
- Introduction to QED

Perturbation theory: If the Hamiltonian can be perturbation expanded (i.e. superposition of smaller and smaller terms)

$\rightarrow$

$$H = H_0 + \lambda V$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots$$

$$|n\rangle = \left|n^{(0)}\right\rangle + \lambda \left|n^{(1)}\right\rangle + \lambda^2 \left|n^{(2)}\right\rangle + \cdots$$

$$\left|n^{(1)}\right\rangle = \sum_{k \neq n} \frac{\left\langle k^{(0)}\right| V \left|n^{(0)}\right\rangle}{E_n^{(0)} - E_k^{(0)}} \left|k^{(0)}\right\rangle .$$

- H_0 unperturbed Hamiltonian with eigenstates $E_n^{(0)}$ and eigenstates $|n^0\rangle$;
- λV 'small perturbation' with eigenstates $E_n^{(1)}$ and eigenstates $|n^1\rangle$ (at first order) + $E_n^{(2\dots)}$ and eigenstates $|n^{2\cdots}\rangle$ (at higher orders);
- λV, λ is between 0 and 1 $\rightarrow$ higher orders count less;
- The term $\langle k^0|V|n^0\rangle$ tells how much V can mix different states;
- the term $E_n^{(0)} - E_k^{(0)}$ indicates how distant is the perturbed state from the unperturbed one.

Interactions by Particle Exchange

If the Hamiltonian can be perturbation expanded
(i.e. superposition of smaller and smaller terms)

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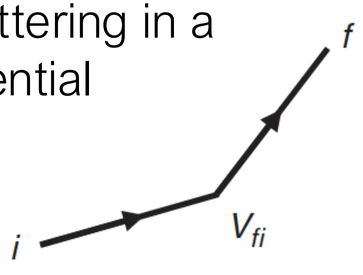
$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

Fermi's Golden rule:
transitions between states

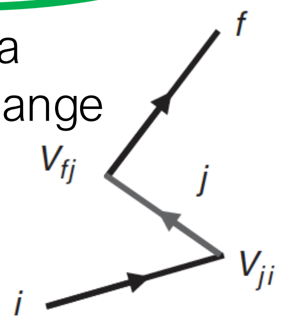
$$T_{fi} = \langle f|V|i\rangle + \sum_{j \neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \cdots .$$

Scattering in a potential



Particles generate potentials, other particles scatter with potential. Unsatisfactory!

Scattering via particle exchange



Particles interact via exchange of particles → no action at 'distance'

Time Ordered Feynman Diagrams

Study reaction $a + b \rightarrow c + d$.

- Exchange of particle X;
- Two possible time orderings.

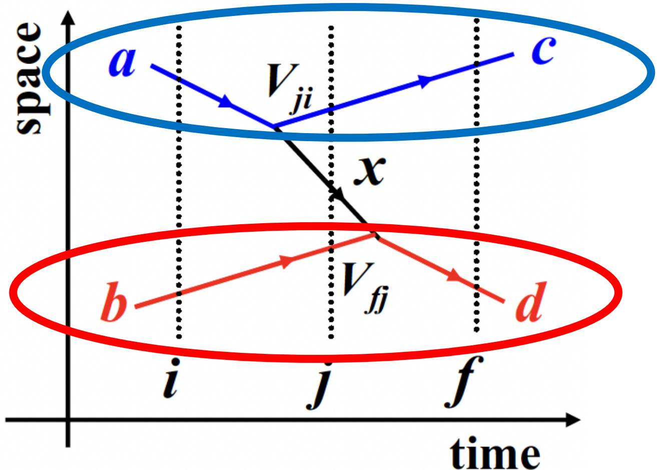
First case:

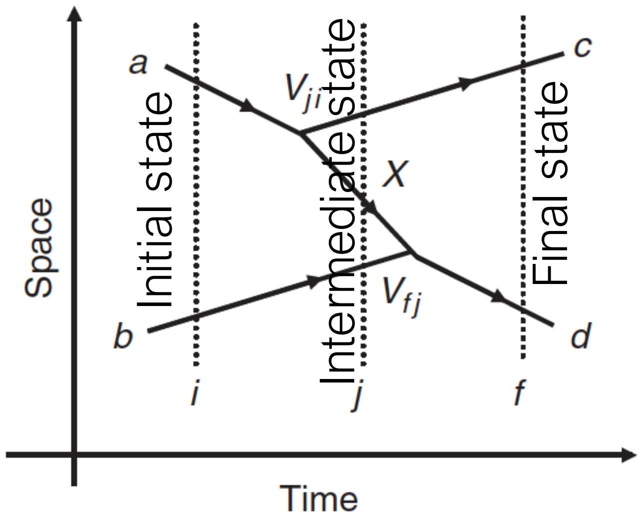
$|i\rangle$ *initial state* $a + b$

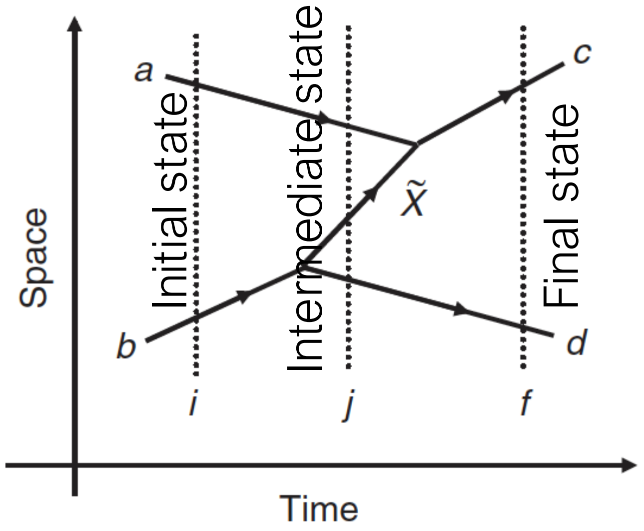
$|j\rangle$ *intermediate state* $c + b + X$

$|f\rangle$ *final state* $c + d$

$\rightarrow a$ (electron) emits X(a photon) that is absorbed by b (a second electron) later





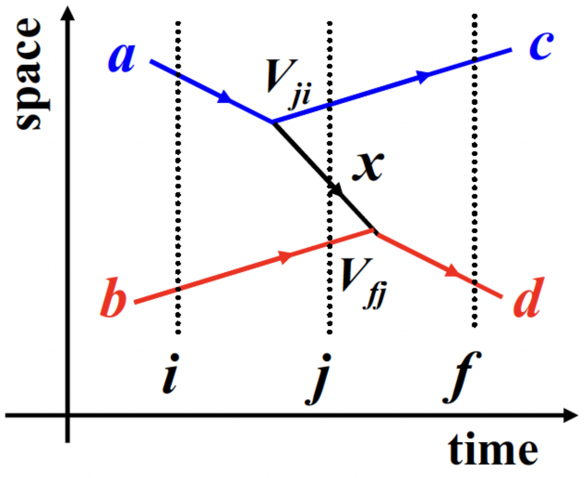


$$T_{fi}^{ab} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} = \frac{\langle d|V|X + b\rangle\langle c + X|V|a\rangle}{(E_a + E_b) - (E_c + E_X + E_b)}$$

Note: the intermediate state $|j\rangle$ has an energy larger than in the initial state: possible for a short period of time

$$\Delta E \Delta t \sim \hbar.$$

Further Analysis -1



V_{ji} non-invariant matrix element
$\mathcal{M}_{ji}$ Lorentz invariant matrix element

$$V_{ji} = \mathcal{M}_{ji} \prod_k (2E_k)^{-1/2}$$

The interaction contains two vertices

- Emission of X;
- Absorption of X.

$$V_{ji} = \langle c + X|V|a\rangle = \frac{\mathcal{M}_{a\to c+X}}{(2E_a 2E_c 2E_X)^{1/2}}$$

Density of states

All particles included in the vertex

Let's assume that $\mathcal{M}_{a\to c+X}$ is the simplest we can think of a simple scalar g_a
That measures the strength of the interaction

$$V_{ji} = \langle c + X|V|a\rangle = \frac{g_a}{(2E_a 2E_c 2E_X)^{1/2}}$$

Same for the second vertex: g_b

$$V_{fj} = \langle d|V|X + b\rangle = \frac{g_b}{(2E_b 2E_d 2E_X)^{1/2}}$$

Further Analysis -2

The final result is then

$$T_{fi}^{ab} = \frac{\langle d|V|X+b\rangle\langle c+X|V|a\rangle}{(E_a + \cancel{E_b}) - (E_c + E_X + \cancel{E_b})} = \frac{1}{2E_X}\cdot\frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}}\cdot\frac{g_a g_b}{(E_a - E_c - E_X)}$$

$$2E_x = \sqrt{2E_x}\cdot\sqrt{2E_x}$$

The Lorentz invariant matrix element $\mathcal{M}_{ji}$ for

$$a + b \to c + d$$

Is related to the transition matrix element (not LI !) T_{fi}^{ab} by

$$\mathcal{M}_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab}.$$

$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X}\cdot\frac{g_a g_b}{(E_a - E_c - E_X)}$$

Similar expression for the 2nd time-ordering

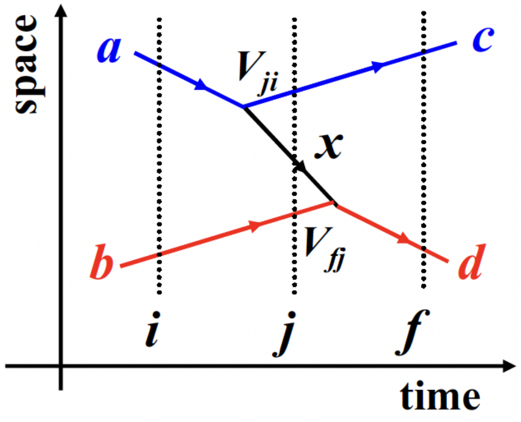
$$\mathcal{M}_{fi}^{ba} = \frac{1}{2E_X}\cdot\frac{g_a g_b}{(E_b - E_d - E_X)}$$

Further Analysis -3

The probability for a given process is the sum of all probabilities of how that process can occur

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{ba}$$

$$= \frac{g_a g_b}{2E_X} \cdot \left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X} \right)$$



If we take into account energy conservation $E_a + E_b = E_c + E_d \rightarrow E_b - E_d = E_c - E_a$

$$\mathcal{M}_{fi} = \frac{g_a g_b}{2E_X} \cdot \left(\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X} \right)$$

$$= \frac{g_a g_b}{(E_a - E_c)^2 - E_X^2}.$$

We observe that:

$$E_x^2 = \boldsymbol{p}_X^2 + m_X^2$$

And that at the 1st vertex

$$p_X = (p_a - p_c)$$

and for the 2nd vertex

$$p_X = (p_b - p_d) = (p_a - p_c)$$

$$\Rightarrow E_x^2 = p_X^2 + m_X^2 = (p_a - p_c)^2 + m_X^2$$

$$\mathcal{M}_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\mathbf{p}_a - \mathbf{p}_c)^2 - m_X^2}$$

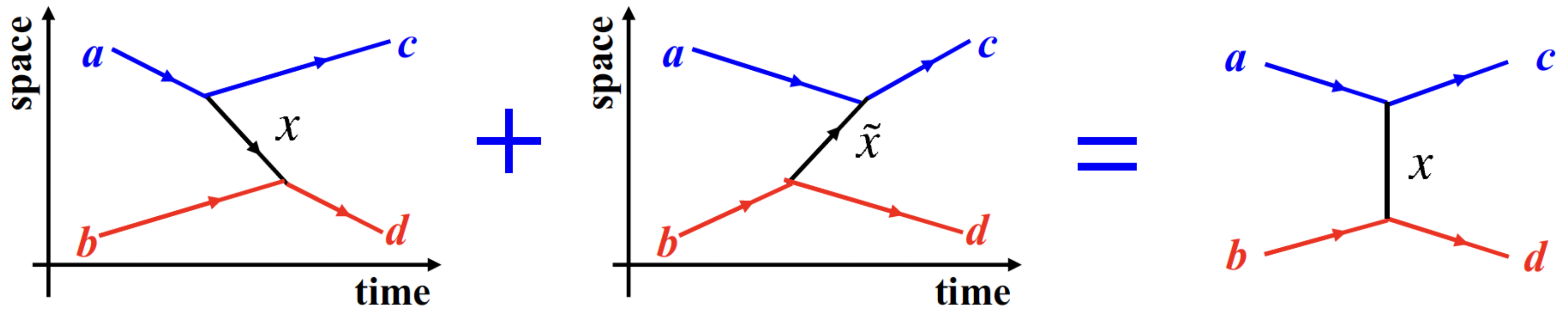
$$= \frac{g_a g_b}{(p_a - p_c)^2 - m_X^2},$$

But since $q = p_a - p_c,$

$$\mathcal{M}_{fi} = \frac{g_a g_b}{q^2 - m_X^2}.$$

Feynman Diagrams

Sum of all possible time orderings, is Lorentz invariant → a frame independent matrix element.



$$\mathcal{M}_{fi} = \frac{g_a g_b}{q^2 - m_X^2}.$$

- Momentum conserved at vertices
- Energy not conserved at vertices
- Exchanged particle "on mass shell"

$$E_X^2 - |\vec{p}_X|^2 = m_X^2$$

- Momentum AND energy conserved at interaction vertices
- Exchanged particle "off mass shell"

$$E_X^2 - |\vec{p}_X|^2 \neq m_X^2$$

→ Virtual Particle

Low energy description of 'scattering of non-relativistic electrons in a potential': the potential V(r) that reproduces low energy data is the Yukawa potential:

$$V(r) = g_a \cdot g_b \cdot e^{-mt}/r$$

For the exchange of a $m = 0$ particle (a photon) → familiar $^1/_r$ Coulomb potential.

Quantum Electrodynamics (QED)

QED is theory of EM interactions.

$$\mathcal{M} = \langle \psi_c | V | \psi_a \rangle \frac{1}{q^2 - m_X^2} \langle \psi_d | V | \psi_b \rangle$$

3 parts:

- The strength of the interaction at each vertex

$$\langle \psi_c | V | \psi_a \rangle \frac{1}{q^2 - m_X^2} \langle \psi_d | V | \psi_b \rangle$$

- The propagator

In the simplest choice of a LI matrix element, we have chosen a scalar interaction

$$\langle \psi_c | V | \psi_a \rangle \propto g_a$$
$$\langle \psi_d | V | \psi_b \rangle \propto g_b$$

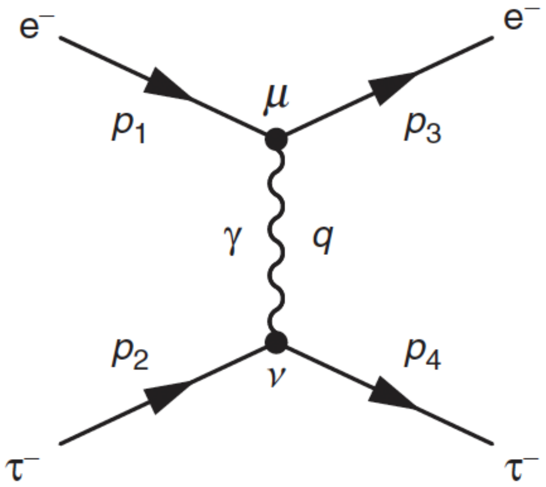
In a realistic treatment of EM interactions, we have to consider that the

photon is a spin 1 particle

→ we need to account for polarisation states

Free photon wavefunction: plane wave + 4-vector for the polarisation:

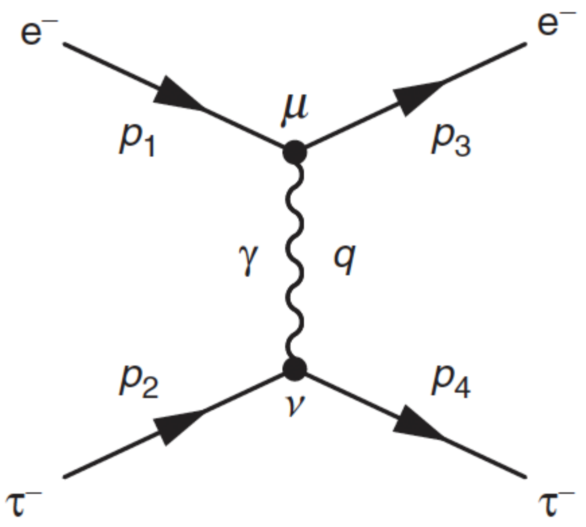
$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}.$$



Interaction Fermion (Charge q) and EM Field

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}.$$

$$e^-\tau^- \to e^-\tau^-$$



ε^μ: 4 vector indicating polarisation

A photon propagating along the z direction has 2 orthogonal polarisation states

$$\varepsilon^{(1)} = (0,1,0,0) \quad \text{and} \quad \varepsilon^{(2)} = (0,0,1,0).$$

Interaction between a fermion with charge q and an EM field $A_\mu(\phi, \boldsymbol{A})$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0,$$

Free Dirac equation

The same substitution we studied for Dirac particles
$A_\mu = (\phi, \boldsymbol{A}), \partial_\mu = (\partial/\partial t, +\boldsymbol{\nabla})$

$$\partial_\mu \to \partial_\mu + iqA_\mu$$

$$\gamma^\mu\partial_\mu\psi + iq\gamma^\mu A_\mu\psi + im\psi = 0.$$

Derive Hamiltonian:

1. Multiply all terms by γ^0 ($\gamma^0\,\gamma^0 = 1$)

$$i\frac{\partial\psi}{\partial t} + i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi - q\gamma^0\gamma^\mu A_\mu\psi - m\gamma^0\psi = 0,$$

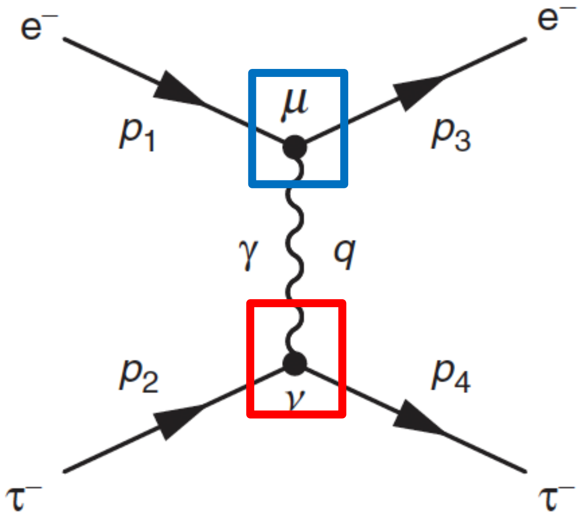
2. Remember

$$\hat{H}\psi = i\frac{\partial\psi}{\partial t},$$

-Hamiltonian

The Hamiltonian Interaction EM

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}.$$



$e^-\tau^- \to e^-\tau^-$

$$i\frac{\partial\psi}{\partial t} + i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi - q\gamma^0\gamma^\mu A_\mu\psi - m\gamma^0\psi = 0,$$

$$\hat{H} = (m\gamma^0 - i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}) + q\gamma^0\gamma^\mu A_\mu$$

Free particle Hamiltonian

$$\hat{V}_D = q\gamma^0\gamma^\mu A_\mu.$$

Interaction EM field with Dirac particle

4-component spinor

μ vertex

$$\langle\psi(p_3)|\hat{V}_D|\psi(p_1)\rangle \to u_e^\dagger(p_3)\, Q_e e\gamma^0\gamma^\mu \varepsilon_\mu^{(\lambda)} u_e(p_1)$$

$e^-(p_3)$ $e^-(p_1)$ $e^-(p_3)$ $e^-(p_1)$

ν vertex

$$u_\tau^\dagger(p_4)\, Q_\tau e\gamma^0\gamma^\nu \varepsilon_\nu^{(\lambda)*} u_\tau(p_2).$$

$\tau^-(p_2)$ $\tau^-(p_4)$

$e^-\tau^- \to e^-\tau^-$

$e^-\tau^- \to e^-\tau^-$

$$\mathcal{M} = \sum_\lambda \left[u_e^\dagger(p_3)Q_e e\gamma^0\gamma^\mu u_e(p_1)\right]\varepsilon_\mu^{(\lambda)}\frac{1}{q^2}\varepsilon_\nu^{(\lambda)*}\left[u_\tau^\dagger(p_4)Q_\tau e\gamma^0\gamma^\nu u_\tau(p_2)\right].$$

Sum over polarisation states of the photon

Use $$\sum_\lambda \varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)*} = -g_{\mu\nu},$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Transition matrix $$\mathcal{M} = -[Q_e e\,\overline{u}_e(p_3)\gamma^\mu u_e(p_1)]\frac{g_{\mu\nu}}{q^2}[Q_\tau e\,\overline{u}_\tau(p_4)\gamma^\nu u_\tau(p_2)].$$

Define currents $$j_e^\mu = \overline{u}_e(p_3)\gamma^\mu u_e(p_1) \quad \text{and} \quad j_\tau^\nu = \overline{u}_\tau(p_4)\gamma^\nu u_\tau(p_2).$$

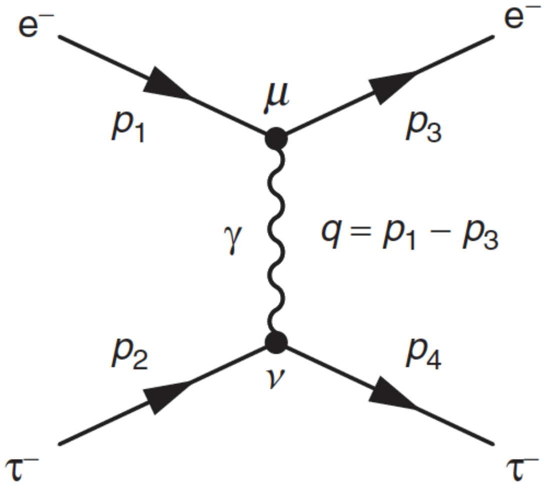
Rewrite in compact form $$\mathcal{M} = -Q_e Q_\tau\, e^2 \frac{j_e \cdot j_\tau}{q^2}$$

Feynman Rules for QED

Three items in Feynman Diagrams

1. Dirac spinors for external fermions (initial and final state particles)
2. A propagator representing the virtual photon

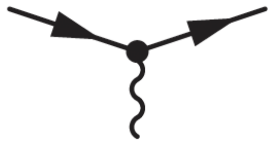
For each item one term; the product of these terms gives $-i\mathcal{M}$



$$\overline{u}(p_3)[ie\gamma^{\mu}]u(p_1)$$

$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\overline{u}(p_4)[ie\gamma^{\nu}]u(p_2)$$

initial-state particle:	$u(p)$	
final-state particle:	$\overline{u}(p)$	
initial-state antiparticle:	$\overline{v}(p)$	
final-state antiparticle:	$v(p)$	
initial-state photon:	$\varepsilon_{\mu}(p)$	
final-state photon:	$\varepsilon^{*}_{\mu}(p)$	
photon propagator:	$-\frac{ig_{\mu\nu}}{q^2}$	
fermion propagator:	$-\frac{i(\gamma^{\mu}q_{\mu} + m)}{q^2 - m^2}$	
QED vertex:	$-iQe\gamma^{\mu}$	

There is no QED vertex connecting more than three particles: 1 photon + 2 charged fermions