March 4th: Dirac Equation

Toni Baroncelli
Haiping Peng
USTC

2026

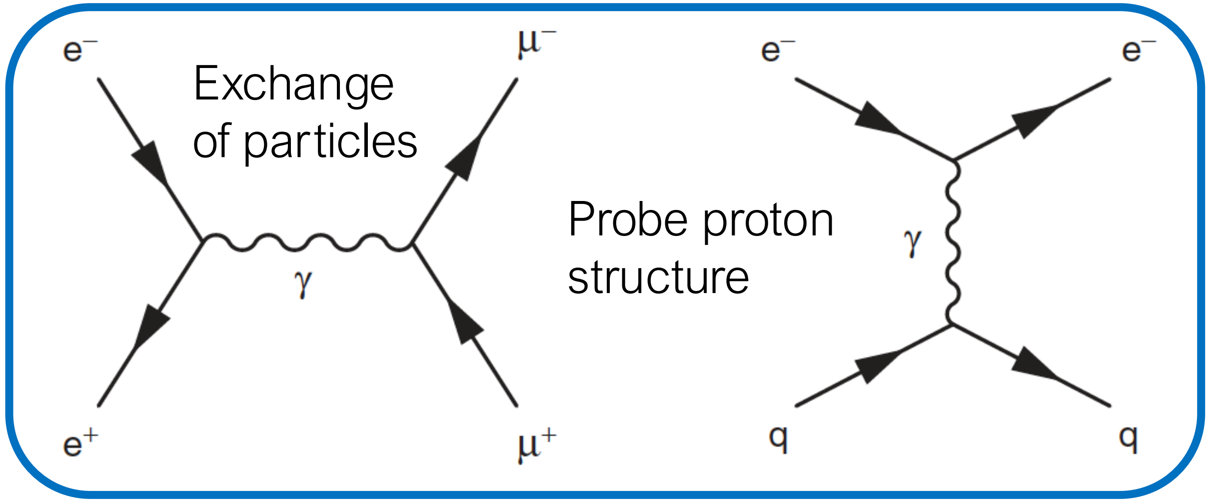
Textual Summary of the Lecture

Today's lecture contains some amount of formalism/apparently complicated technicalities (***review them in the book***). Below a summary in words of the physics points we discuss:

1. Go from a non-relativistic wave function (***Schrödinger***) → a relativistic one, ***Klein-Gordon*** (with difficulties) → a relativistic which fits experimental data (***Dirac***); some technicalities …
2. Elaborate on the Dirac wave function: constraints on the form of the equation from the relativistic invariance;
3. Non-conservation of angular momentum $\boldsymbol{L}$;
4. Solutions of the Dirac equation for a particle at rest;
5. Appearance of the 'intrinsic angular momentum', spin $\boldsymbol{S}$; conservation of $\boldsymbol{L+S}$;
6. Understanding negative energy solutions, antiparticles;
7. Charge conjugation;
8. Parity & Helicity.

The explicit definition of the solutions of the Dirac equation will allow us to do next step:

Computing decay & interaction rates in the 'particle exchange framework'.



From Schrödinger to Klein-Gordon to Dirac

Basic requirement of relativistic Particle Physics : Lorentz invariance of the associated wave-function

History: from Schrödinger to Klein-Gordon to Dirac

Non-relativistic formulation: $E = \frac{\mathbf{p}^2}{2m}.$ Schrödinger equation, obviously non invariant for Lorentz transformations (1st order in E, 2nd order in p)

Relativistic formulation (start of Klein-Gordon): $E^2 = \mathbf{p}^2 + m^2,$ 2nd order in both time and space

Using energy and momentum *operators* $\hat{\mathbf{p}} = -i\nabla$ and $\hat{E} = i\frac{\partial}{\partial t},$ transforms into:

$$\hat{E}^2\psi(\mathbf{x},t) = \hat{\mathbf{p}}^2\psi(\mathbf{x},t) + m^2\psi(\mathbf{x},t). \Rightarrow (\partial^\mu\partial_\mu + m^2)\psi = 0, \quad \partial^\mu\partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

When applied to $\psi(\mathbf{x},t) = Ne^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$, gives $E^2\psi = \mathrm{p}^2\psi + m^2\psi,$ $\Rightarrow$ Energy-momentum relationship OK

But negative energy solutions! $E = \pm\sqrt{\mathrm{p}^2 + m^2}.$

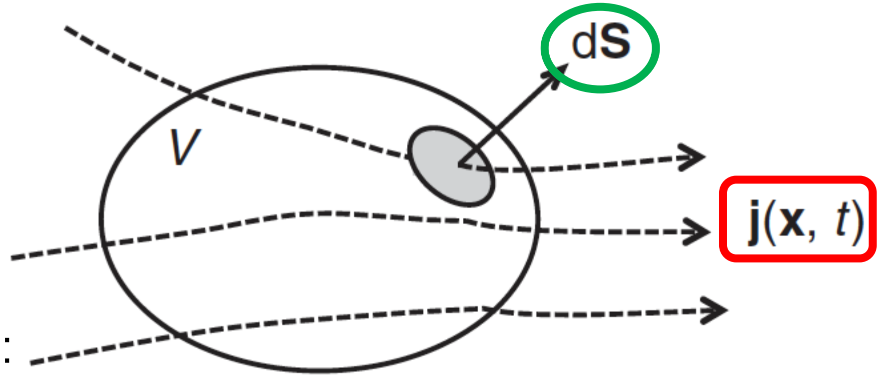
Klein-Gordon: Negative Energy Solutions & More

The physical interpretation of the wave-function is that

$$\psi^*(\boldsymbol{x},t)\psi(\boldsymbol{x},y)d^3\boldsymbol{x}$$

is the probability of finding the particle it represents in a volume $d^3\boldsymbol{x}$.

Introduce probability ***density*** (→probability per unit volume → divide by $d^3\boldsymbol{x}$):



If the particle doesn't decay or doesn't interact → the probability stays constant.

Define probability density: $\rho(x,t) = \psi^*(\boldsymbol{x},t)\psi(\boldsymbol{x},y)d^3\boldsymbol{x}\ /d^3\boldsymbol{x} = \psi^*(\boldsymbol{x},t)\psi(\boldsymbol{x},y)$

Quantify the variation of probability = flux $j(\boldsymbol{x},t)$ of probability leaving the volume V through $d\mathbf{S}$ as

$$\frac{\partial}{\partial t}\int_V \rho(x,t)dV = \int_S j(\boldsymbol{x},t)d\boldsymbol{S}$$

It can be shown that → continuity equation $\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0.$

Reminder:
Klein-Gordon equation
$(\partial^\mu\partial_\mu + m^2)\psi = 0,$

Probability Density & Probability Current

Klein-Gordon equation $$\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - m^2\psi.$$

$$\psi^* \times (4.1) - \psi \times (4.1)^*$$

Compute difference

$$\psi^*\frac{\partial^2\psi}{\partial t^2} - \psi\frac{\partial^2\psi^*}{\partial t^2} = \psi^*(\nabla^2\psi - m^2\psi) - \psi(\nabla^2\psi^* - m^2\psi^*)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) = \nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*).$$

Probability current

$$\rho(x,t) = \psi^*(\boldsymbol{x},t)\psi(\boldsymbol{x},y)$$

continuity equation

$$\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0.$$

And compare to continuity equation

$$\psi^*\frac{\partial^2\psi}{\partial t^2} - \psi\frac{\partial^2\psi^*}{\partial t^2} = \psi^*(\nabla^2\psi - m^2\psi) - \psi(\nabla^2\psi^* - m^2\psi^*)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) = \nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*).$$

$$\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0.$$

identify $\rho = i\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)$ and $\mathbf{j} = -i(\psi^*\nabla\psi - \psi\nabla\psi^*).$

where the factor of i is included to ensure that the probability density is real.

Probability Density & Probability Current

that applied to a plane wave solution

$$\psi(\mathbf{x},t) = Ne^{i(\mathbf{p}\cdot\mathbf{x}-Et)},$$

gives

$$\rho = 2|N|^2 E \quad \text{and} \quad \mathbf{j} = 2|N|^2 \mathbf{p},$$

$\rho = 2|N|^2 E$

- Probability density goes like E → relativistic length contraction
- Negative energy solution ⇒ negative probability ⇒ impossible, unphysical

Dirac equation: both negative energy solutions and description of spin of particles

The Dirac Equation

Try writing an equation 1st order in both time and space → Dirac Equation
And fulfill energy-momentum Einstein relation

Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\psi = 0,$$

Dirac

$$\hat{E}\psi = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)\psi, \qquad i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)\psi.$$

What are constants $\boldsymbol{\alpha}, \beta$??

Dirac equation must also satisfy Klein-Gordon equation!

$$-\frac{\partial^2 \psi}{\partial t^2} = \left(i\alpha_x \frac{\partial}{\partial x} + i\alpha_y \frac{\partial}{\partial y} + i\alpha_z \frac{\partial}{\partial z} - \beta m\right)\left(i\alpha_x \frac{\partial}{\partial x} + i\alpha_y \frac{\partial}{\partial y} + i\alpha_z \frac{\partial}{\partial z} - \beta m\right)\psi,$$

$$\frac{\partial^2 \psi}{\partial t^2} = \alpha_x^2 \frac{\partial^2 \psi}{\partial x^2} + \alpha_y^2 \frac{\partial^2 \psi}{\partial y^2} + \alpha_z^2 \frac{\partial^2 \psi}{\partial z^2} - \beta^2 m^2 \psi$$

Looks like Klein-Gordon

$$+ (\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2 \psi}{\partial x \partial y} + (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2 \psi}{\partial y \partial z} + (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2 \psi}{\partial z \partial x}$$

$$+ i(\alpha_x\beta + \beta\alpha_x)m\frac{\partial \psi}{\partial x} + i(\alpha_y\beta + \beta\alpha_y)m\frac{\partial \psi}{\partial y} + i(\alpha_z\beta + \beta\alpha_z)m\frac{\partial \psi}{\partial z}.$$

Klein-Gordon

α and β cannot be numbers → matrices

The Dirac Equation: α and β

For Dirac to satisfy Klein-Gordon then

$\boldsymbol{\alpha}, \beta$ *cannot be numbers*

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I,$$
$$\alpha_j \beta + \beta \alpha_j = 0,$$
$$\alpha_j \alpha_k + \alpha_k \alpha_j = 0 \quad (j \neq k),$$

α and β are 4 mutually anticommuting matrices

Lowest dimension: 4x4

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The Dirac equation is a 4x4 matrix of operators that act on a ***four-component wave-function*** (Dirac spinor) 4 degrees of freedom

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

If all particles were massless then the β term would not be needed $\hat{E}\psi = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m)\psi,$

Particles would be described by a two-component object (Weyl spinor)

Dirac: Negative Probability Density?

As we did for Klein-Gordon compute difference between

A Hermitian matrix is a square matrix that is equal to the transpose of its conjugate matrix.

$$\begin{bmatrix} 3 & 3+i \\ 3-i & 2 \end{bmatrix} \xrightarrow{\text{conjugate}} \begin{bmatrix} 3 & 3-i \\ 3+i & 2 \end{bmatrix} \xrightarrow{\text{Transpose}} \begin{bmatrix} 3 & 3+i \\ 3-i & 2 \end{bmatrix}$$

$$-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + m\beta\psi = +i\frac{\partial \psi}{\partial t},$$

Wave function

$$+i\frac{\partial \psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial \psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial \psi^\dagger}{\partial z}\alpha_z^\dagger + m\psi^\dagger\beta^\dagger = -i\frac{\partial \psi^\dagger}{\partial t}.$$

Hermitian conjugate Wave function

And compare to the continuity equation (omitting calculation → book)

Probability density = $\rho = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$ ⇒ positive by definition

- Dirac formulation gives positive defined probability density;
- Dirac particles are more complex than Klein Gordon ones: four components wavefunctions
  - Additional degrees of freedom (spin, intrinsic angular momentum);
  - Can be shown to describe particles & antiparticles

Pauli Matrices

Hermitian matrix is a complex square matrix that is equal to its own conjugate transpose

$\boldsymbol{A}$ is Hermitian $\Leftrightarrow a_{ij} = \overline{aji}$ $\boldsymbol{A}$ is Hermitian $\Leftrightarrow A = \overline{A^T}$

The Pauli matrices are a set of three 2 ×2 complex matrices that are traceless, Hermitian, unitary.

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dirac Particles and Angular Momentum

Reminder: the time dependence of an observable $\hat{O}$ is given by

$$\frac{dO}{dt} = \frac{d}{dt}\langle \hat{O} \rangle = i\langle \psi | [\hat{H}, \hat{O}] | \psi \rangle.$$

if $\frac{d\hat{O}}{dt} = 0$ the observable is conserved $\leftrightarrow$ $[\hat{H}, \hat{O}] = 0$
$\rightarrow$ the two operators commute

Angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = (yp_z - zp_y, zp_x - xp_z, xp_y - yp_x).$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \quad \text{and} \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x.$$

$$[\alpha_x\hat{p}_x + \alpha_y\hat{p}_y + \alpha_z\hat{p}_z,\ \hat{y}\hat{p}_z - \hat{z}\hat{p}_y]$$

$$[A, BC] = [A, B]C + B[A, C]$$

Schrödinger equation (*non relativistic*)

$$\hat{H}_{SE} = \frac{\hat{\mathbf{p}}^2}{2m}, \quad [\widehat{H_{SE}}, \hat{L}] = 0$$ $\rightarrow$. Angular Momentum is conserved

Dirac equation (*relativistic*)

$$\hat{H}_D = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m, \quad [\hat{H}_D, \hat{\mathbf{L}}] = [\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m, \hat{\mathbf{r}} \times \hat{\mathbf{p}}] = [\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}, \hat{\mathbf{r}} \times \hat{\mathbf{p}}].$$

$$[\hat{H}_D, \hat{\mathbf{L}}] = -i\boldsymbol{\alpha} \times \hat{\mathbf{p}}.$$

Angular Momentum is NOT conserved

Introduce a new operator:

$$\hat{\mathbf{S}} \equiv \tfrac{1}{2}\hat{\boldsymbol{\Sigma}} \equiv \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

$$\hat{\Sigma}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{\Sigma}_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \text{and} \quad \hat{\Sigma}_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Spin & Angular Momentum of Dirac Particles

It can be shown that also $\hat{S}$ doesn't commute with Hamiltonian $[\hat{H}_D, \hat{\mathbf{S}}] = i\alpha \times \hat{\mathbf{p}}.$

We have just seen that $[\hat{H}_D, \hat{\mathbf{L}}] = -i\alpha \times \hat{\mathbf{p}}.$

- It is natural to associate the operator $\hat{S}$ with the intrinsic angular momentum of the particle;

Which translates into $\left[\hat{H}_D, \hat{\mathbf{J}}\right] \equiv \left[\hat{H}_D, \hat{\mathbf{L}} + \hat{\mathbf{S}}\right] = 0.$

- The total angular momentum $\hat{L} + \hat{S}$ is a conserved quantity;
- Dirac particles have all intrinsic angular momentum $s = \frac{1}{2}$;
- The intrinsic magnetic moment of a Dirac particle is $\hat{\mu} = \frac{q}{m}\hat{\mathbf{S}},$ where q *and* m are the charge and the mass of the Dirac particle

Dirac equation includes naturally the description of spin ½ particles.
This is NOT a mathematical consequence.
This is the consequence of requiring the wavefunction to satisfy a particular structure of the Dirac equation

Covariant Form of the Dirac Equation

The Dirac equation can be expressed in a covariant form (a few steps …)

1. Start from the standard equation $i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi.$
2. Multiply it by β $\quad i\beta\alpha_x\frac{\partial\psi}{\partial x} + i\beta\alpha_y\frac{\partial\psi}{\partial y} + i\beta\alpha_z\frac{\partial\psi}{\partial z} + i\beta\frac{\partial\psi}{\partial t} - \beta^2 m\psi = 0.$
3. Define $\gamma^0 \equiv \beta, \quad \gamma^1 \equiv \beta\alpha_x, \quad \gamma^2 \equiv \beta\alpha_y \quad \text{and} \quad \gamma^3 \equiv \beta\alpha_z,$
4. And $\partial_\mu \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
5. You can rewrite the Dirac equation as $(i\gamma^\mu\partial_\mu - m)\psi = 0,$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Solutions of the Dirac Equation

Which is the physical meaning of the solutions of the Dirac equation? $(i\gamma^\mu \partial_\mu - m)\psi = 0.$

Free particle wavefunctions of spin ½ particles

- $u(E, \boldsymbol{p})$ is 4-component spinor
- No position and time dependence

$$\psi(\mathbf{x}, t) = u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)},$$

Derivatives $\partial_\mu \psi(\boldsymbol{x}, t)$ act only on the exponent

$$\partial_\mu \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\partial_0\psi \equiv \frac{\partial\psi}{\partial t} = -iE\psi, \quad \partial_1\psi \equiv \frac{\partial\psi}{\partial x} = ip_x\psi, \quad \partial_2\psi = ip_y\psi \quad \text{and} \quad \partial_3\psi = ip_z\psi.$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} = 0,$$

$$(\gamma^\mu p_\mu - m)\, u = 0,$$

→ the expression doesn't contain derivatives
It is the free-particle Dirac equation for the spinor $u(E, \boldsymbol{p})$

Dirac Equation: Solution for a Particle at Rest

Nonrelativistic case!

$$\psi(\mathbf{x},t) = u(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)},$$ a particle at rest $$\psi = u(E,0)e^{-iEt},$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} = 0,$$

$$E\gamma^0 u = mu$$

$$E\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{\gamma^0}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}.$$

γ^0 is diagonal $\Leftrightarrow$ 4 orthogonal solutions

The 4 states are also eigenstates of the $\hat{S}_z$ operator

N wavefunction normalisation

Spin up

$$u_1(E,0) = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2(E,0) = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

Spin down

Positive energy solution $E = +m$

Spin up

Spin down

$$u_3(E,0) = N\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_4(E,0) = N\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

Negative energy solution $E = -m$

Particle at Rest: Dirac solution

$$\psi_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt},\ \psi_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt},\ \psi_3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt} \text{ and } \psi_4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}.$$

- Dirac equation for a particle at rest has positive probability density;
- Represents well spinors with spin up and spin down;

- Has still not solved the problem with negative energy solutions!

Dirac Equation: Solution for a Free-Particle

Dirac equation for a free-particle spinor (was $E\gamma^0 u = mu$ for a particle at rest)

Written in full → $(E\gamma^0 - p_x\gamma^1 - p_u\gamma^2 - p_z\gamma^3 - m)u = 0.$

γ^0 $\gamma^{1,2,3}$

$$\left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] u = 0,$$

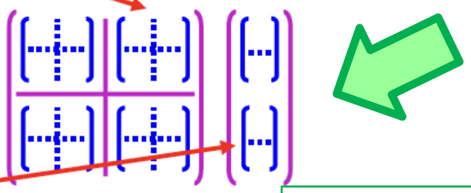
$$= \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -(E+m)I \end{pmatrix}$$

4 solutions !

Note $\vec{\sigma}.\vec{p} = p_x\sigma_x + p_y\sigma_y + p_z\sigma_z$

Note in the above equation the 4x4 matrix is written in terms of four 2x2 sub-matrices

- **Writing the four component spinor as** $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$



Pauli matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

u_A^1
u_A^2
u_B^1
u_B^2

Describe u as sum of u_a and u_b

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\boldsymbol{\sigma}\cdot\mathbf{p} \equiv \sigma_x p_x + \sigma_y p_y + \sigma_z p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}.$$

Explicit Positive & Negative Energy Solutions

$$\begin{pmatrix} (E-m)I & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -(E+m)I \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$\vec{\sigma}.\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

$$\vec{\sigma}.\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Gives u_a as a function of u_b

$$u_A = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E-m} u_B,$$

$$u_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_A.$$

One more step: explicit u_a: u_A^1 and u_A^2.
(2 solutions for positive energy)

$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Simplest Orthogonal choice

The corresponding $u_B^{1,2}$ terms can be derived as

$$u_B = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_A = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

Toni Baroncelli: Introduction to Particle Physics

Explicit Positive & Negative Energy Solutions

u_A^1
u_A^2
u_B^1
u_B^2

$$\begin{pmatrix} 1 \\ 0 \\ u_B^1 = f(u_A^1) \\ u_B^2 = f(u_A^2) \end{pmatrix}$$

The first two solutions of the Dirac equation for a free particle.

Positive or negative energy?

$$u_1(E,\mathbf{p}) = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(E,\mathbf{p}) = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix},$$

Compare with solutions for a particle at rest:
→ the spin operator $\hat{S}$ doesn't return 0 or 1

$$u_1(E,0) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2(E,0) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{\Sigma}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

X-component of spin operator

Other two solution are obtained with

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And derive u_a from u_b

$$u_A = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E - m} u_B,$$

The choices are arbitrary; just like choosing one reference frame.
It is the simplest choice!

Explicit Positive & Negative Energy Solutions

- Explicitly write down 4 solutions;

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}.$$

- Which energy do they correspond to? All these solutions satisfy Dirac equation:

$$\psi_i = u_i(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$$

If you put back any of these solutions into Dirac equation → get $E^2 = p^2 + m^2$

If you put $p = 0$ then you get

$$\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$$

Spinors $u_{1,2}$ reduce to the positive energy solution of a Dirac particle at rest $\quad E = +\left|\sqrt{p^2 + m^2}\right|$

And the same for $u_{3,4}$ $\quad E = -\left|\sqrt{p^2 + m^2}\right|$

- There are 4 independent solutions;
- We cannot avoid negative energy solutions

Where Did We Get? Where Did We Start From?

Did we derive laws of nature from algebra of Pauli (or Gamma or Whatever) matrices?

NO!

We made use of a handful of basic ideas based on experimental facts:

- The probability of finding ONE particle (not decaying and not interacting is ONE;
- → Quantum Mechanics;
- Fermi exclusion principle
- Laws of nature do not depend on the reference frame where you observe them;
- → Lorentz transformations
- → $E^2 = p^2 + m^2$

} Nature

These ingredients have been elaborated, expanded, represented using algebra & operators & Co to give a mathematical representation of nature. Nature is not in this representation, Nature is in the list of points above

} Technique

Antiparticles & Negative Energy Solutions

Dirac equation:

- Incredibly good framework for spin ½ particles;
- Spin and magnetic moments emerge naturally;
- Negative energy solutions cannot be excluded as '*unphysical*';
- Must provide a '*physical*' interpretation for these solutions.

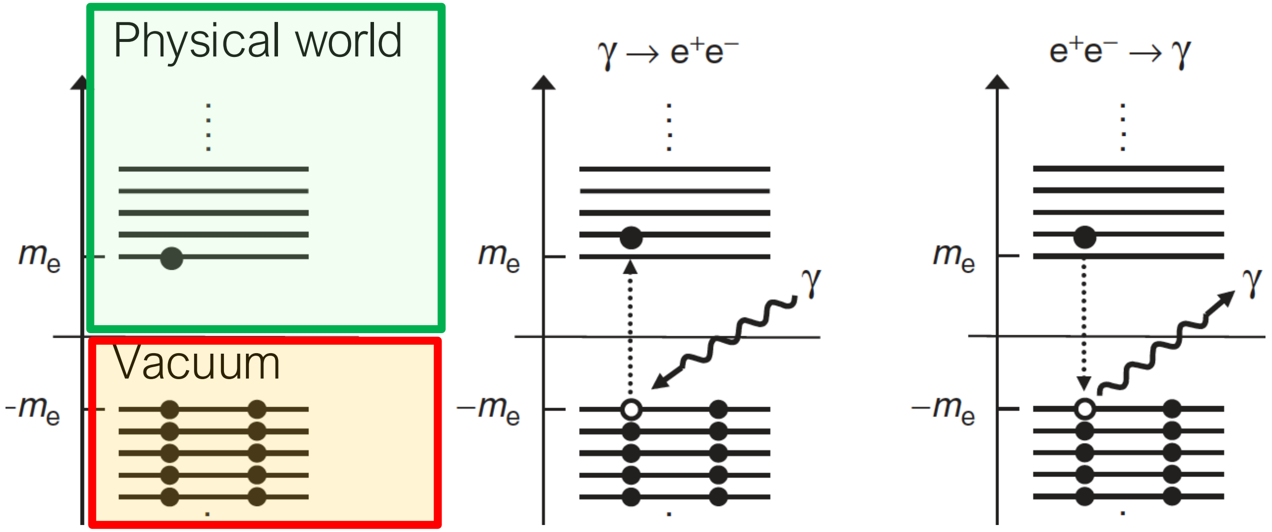
Difficulty:

- If really '***negative energy states***' existed, and were ***accessible***, then all positive energy electrons would fall into this lower energy states;

First attempt: the 'Dirac' sea

The vacuum is fully occupied by negative energy states

- → no hole is present for +energy electrons to go;
- → '***negative energy states***' are inaccessible
- Fermi exclusion principle prevents electrons from occupying the same position/energy



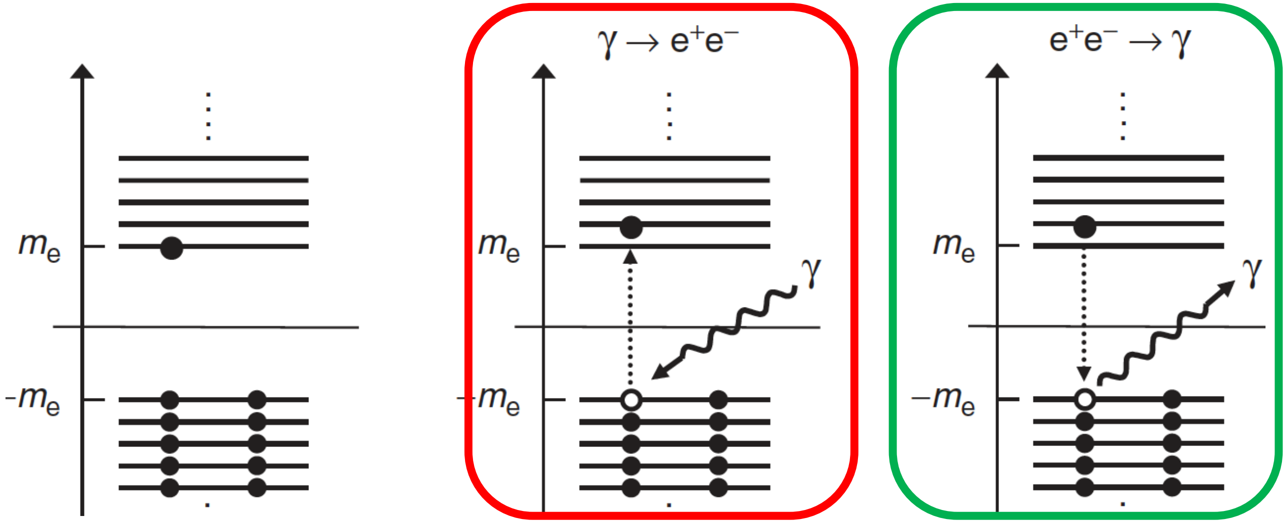
The Dirac see

This idea seemed not bad:

- If a photon, energy $> 2m_e$ excites one 'negative energy electron' would leave a 'hole' with
  - Less negative energy;
  - A loss of charge -1 → charge +1
- A positive energy *electron* with charge +1
- Pair creation

- A positive energy electron falling into one available 'hole' would give
  - Disappearance of energy (negative energy);
  - Disappearance of a charge -1 (charge +1)
- Electron/positron annihilation.

→ **antiparticles ??!!**



Difficulties:

- The Dirac see would be populated by an infinite number of antielectrons → infinite energy! How to handle this?
- Today we know that also anti-bosons exist and the Fermi exclusion principle would not exclude occupying the same 'hole'

The Feynman–Stückelberg interpretation

We know today that:
Each spin ½ fermion has a spin ½ partner with exactly same characteristics BUT opposite charge.

Solution:

Negative energy fermions that propagate backward in time $\equiv$ Positive energy antifermions that propagate forward in time

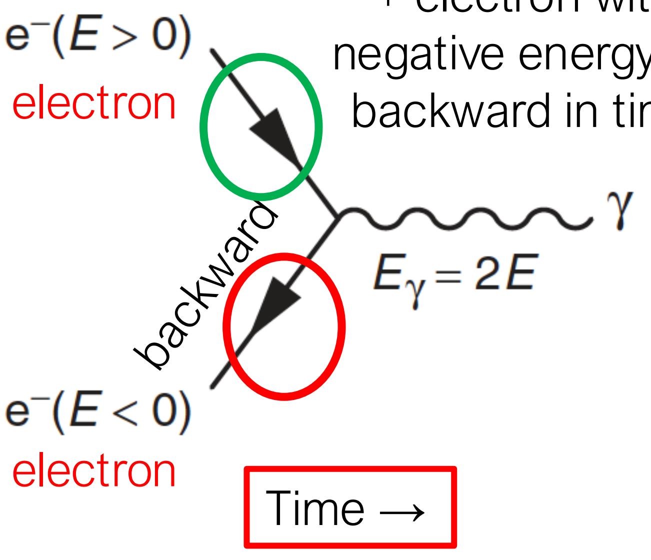
Wavefunction doesn't change when
$(E) \rightarrow (-E) + (-t) \rightarrow (-t)$

$$e^{[-iEt]} \equiv e^{[-i(-E)(-t)]}$$

Graphic convention:

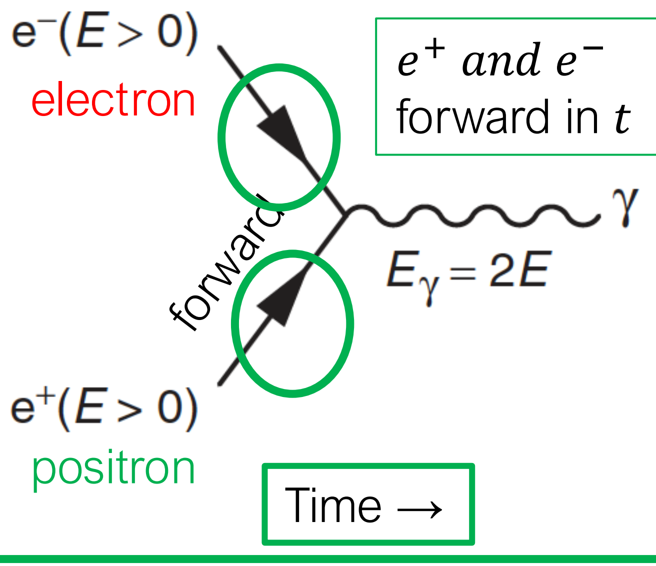
Antiparticles are drawn as travelling back in time

electron emits γ (2E) + electron with negative energy → backward in time



electron (+E) annihilates with positron (+E) and emits γ

e^+ and e^- forward in t



Physical Antiparticle Spinors

Use 'physical spinors': physical energy and momentum: go from

$u_3 \to v_1$ and $u_4 \to v_2$

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}.$$

$$v_1(E,\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} = u_4(-E,-\mathbf{p})e^{i[-\mathbf{p}\cdot\mathbf{x}-(-E)t]}$$

$$v_2(E,\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)} = u_3(-E,-\mathbf{p})e^{i[-\mathbf{p}\cdot\mathbf{x}-(-E)t]}.$$

$N_i = \sqrt{E+m}$ wavefunction normalisation (Lorentz contraction) to give $2E$ particles per unit volume

Same procedure as for u_1 and u_2

Dirac particle solution $\psi_i = u_i e^{+i(\mathbf{p}\cdot\mathbf{x}-Et)}$

$$u_1(p) = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(p) = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix},$$

Dirac antiparticle solution $\psi_i = v_i e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}$

$E = +\left|\sqrt{\mathbf{p}^2+m^2}\right|$

$$v_1(p) = \sqrt{E+m} \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2(p) = \sqrt{E+m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}.$$

Operators Acting on Antiparticles

Operators that return physical energy and momentum of antiparticles have to be modified (to return physical energy and momentum):

$$\hat{H}^{(v)} = -i\frac{\partial}{\partial t} \quad \text{and} \quad \hat{\mathbf{p}}^{(v)} = +i\boldsymbol{\nabla},$$

Feynman–Stückelberg interpretation: $(E, \mathbf{p}) \rightarrow (-E, -\mathbf{p})$.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \rightarrow -\mathbf{L}.$$

To maintain $\left[\widehat{H_D}, \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}\right] = 0$ for antiparticles $\quad \hat{\mathbf{S}}^{(v)} = -\hat{\mathbf{S}},$

Dirac sea picture: a spin-up hole in the negative energy Dirac sea, leaves the vacuum in a net spin-down state.

Charge Conjugation

Symmetries are very important in Particle Physics. We will discuss more

Charge conjugation is a discrete transformation of particles into antiparticles

- Classical dynamics → how the Charge conjugation operator is defined.
- Motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \boldsymbol{A})$

$$E \to E - q\phi \quad \text{and} \quad \mathbf{p} \to \mathbf{p} - q\mathbf{A},$$

4 vector notation $p_\mu \to p_\mu - qA_\mu.$

$\phi, \boldsymbol{A}$ scalar and vector potentials, q is the charge of the particle

Classical Physics → Quantum Mechanics $\hat{E} = i\partial/\partial t, \quad \hat{\mathbf{p}} = -i\boldsymbol{\nabla}$ ⇒ $i\partial_\mu \to i\partial_\mu - qA_\mu.$

Dirac equation motion of a charged *particle* $q = -e$ in an EM field becomes

$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\psi = 0.$$

particle

Free particle Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0,$$

Dirac equation motion of a charged *antiparticle* $q = +e$ in an EM field becomes

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi' + im\psi' = 0.$$

antiparticle

$\psi' = i\gamma^2\psi^*,$ ⇒ $\psi' = \hat{C}\psi = i\gamma^2\psi^*.$

Let's study spin and helicity

Z-component of the spin operator

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$u_1(E,0) = N\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2(E,0) = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix};$$

Particle at rest spinors *(E>0)*
Eigenstates of $\hat{S}_z$

Clearly not true for $p \neq 0$. In general NOT eigenstates

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad \text{and} \quad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix},$$

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}.$$

In the special case $p_z = \pm p, p_{x,y} = 0$.

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix}, \; u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp p}{E+m} \end{pmatrix}, \; v_1 = N\begin{pmatrix} 0 \\ \frac{\mp p}{E+m} \\ 0 \\ 1 \end{pmatrix} \text{ and } v_2 = N\begin{pmatrix} \frac{\pm p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Spin Effect

$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \tfrac{1}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},$$

$$\hat{\mathbf{S}}^{(v)} = -\hat{\mathbf{S}},$$

The action of $\widehat{S_Z}$ $(particles)$

$$\hat{S}_z u_1(E,0,0,\pm p) = +\tfrac{1}{2}u_1(E,0,0,\pm p)$$ Spin up

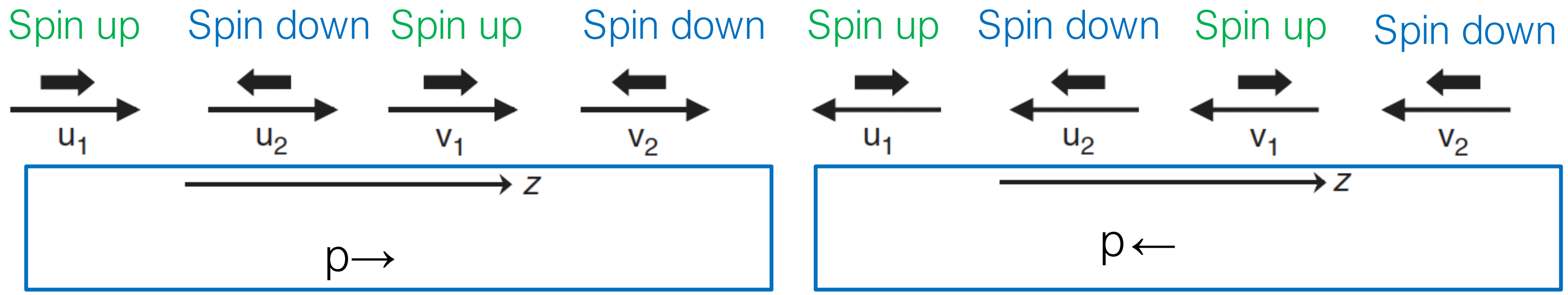
$$\hat{S}_z u_2(E,0,0,\pm p) = -\tfrac{1}{2}u_2(E,0,0,\pm p)$$ Spin down

and of $\widehat{S_Z^v}$ $(antiparticles)$

$$\hat{S}_z^{(v)} v_1(E,0,0,\pm p) \equiv -\hat{S}_z v_1(E,0,0,\pm p) = +\tfrac{1}{2}v_1(E,0,0,\pm p)$$ Spin up

$$\hat{S}_z^{(v)} v_2(E,0,0,\pm p) \equiv -\hat{S}_z v_2(E,0,0,\pm p) = -\tfrac{1}{2}v_2(E,0,0,\pm p)$$ Spin down

Possible configurations:

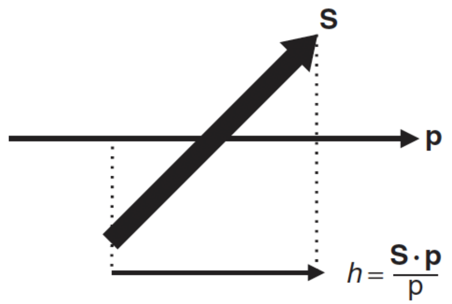


Helicity

- Cross sections calculation depends on spin states;
- The z component of the Spin operator is of limited use;
- The z component of the Spin operator does not commute with the Dirac Hamiltonian;

→ introduce

$$h \equiv \frac{\mathbf{S}\cdot\mathbf{p}}{\mathrm{p}}.$$



Helicity: projection of the spin along the direction of motion

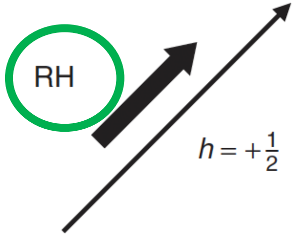
The corresponding helicity operator is $\hat{h} = \frac{\hat{\boldsymbol{\Sigma}}\cdot\hat{\mathbf{p}}}{2\mathrm{p}} = \frac{1}{2\mathrm{p}}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}$

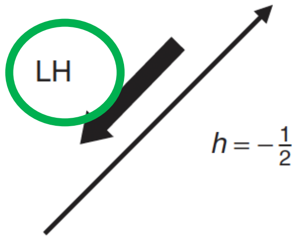
It can be shown that the Dirac Hamiltonian commutes with $\hat{h}$ $[H_D, \hat{\boldsymbol{\Sigma}}\cdot\hat{\mathbf{p}}] = 0$

simultaneous eigenstates of the free particle Dirac Hamiltonian and the helicity operator.

For a fermion the eigenvalues of the helicity operator ±1/2. These states called

- right-handed and
- left-handed helicity states





$$\hat{h}u = \lambda u.$$

Helicity operator acting on a spinor → returns λ

Helicity Eigenstates

Need explicit solutions of Dirac Equations that are also eigenstates of Helicity

$$\frac{1}{2\mathrm{p}}\begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbf{p} & 0\\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p}\end{pmatrix}\begin{pmatrix}u_A\\ u_B\end{pmatrix}=\lambda\begin{pmatrix}u_A\\ u_B\end{pmatrix}$$

$$\mathbf{p}=(\mathrm{p}\sin\theta\cos\phi,\ \mathrm{p}\sin\theta\sin\phi,\ \mathrm{p}\cos\theta),$$

$$(\boldsymbol{\sigma}\cdot\mathbf{p})u_A=2\mathrm{p}\,\lambda u_A,$$
$$(\boldsymbol{\sigma}\cdot\mathbf{p})u_B=2\mathrm{p}\,\lambda u_B.$$

$$\frac{1}{2\mathrm{p}}(\boldsymbol{\sigma}\cdot\mathbf{p})=\frac{1}{2\mathrm{p}}\begin{pmatrix}p_z & p_x-ip_y\\ p_x+ip_y & -p_z\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}.$$

$s=\sin\left(\frac{\theta}{2}\right)$ and $c=\cos\left(\frac{\theta}{2}\right)$

Particles, right-handed spinor $u_\uparrow$ left-handed spinor $u_\downarrow$

$$u_\uparrow=\sqrt{E+m}\begin{pmatrix}c\\ se^{i\phi}\\ \frac{\mathrm{p}}{E+m}c\\ \frac{\mathrm{p}}{E+m}se^{i\phi}\end{pmatrix}\quad u_\downarrow=\sqrt{E+m}\begin{pmatrix}-s\\ ce^{i\phi}\\ \frac{\mathrm{p}}{E+m}s\\ -\frac{\mathrm{p}}{E+m}ce^{i\phi}\end{pmatrix},$$

Antiparticles, right/left-handed spinor $v_\uparrow$ $v_\downarrow$

$$v_\uparrow=\sqrt{E+m}\begin{pmatrix}\frac{\mathrm{p}}{E+m}s\\ -\frac{\mathrm{p}}{E+m}ce^{i\phi}\\ -s\\ ce^{i\phi}\end{pmatrix}\quad v_\downarrow=\sqrt{E+m}\begin{pmatrix}\frac{\mathrm{p}}{E+m}c\\ \frac{\mathrm{p}}{E+m}se^{i\phi}\\ c\\ se^{i\phi}\end{pmatrix}.$$

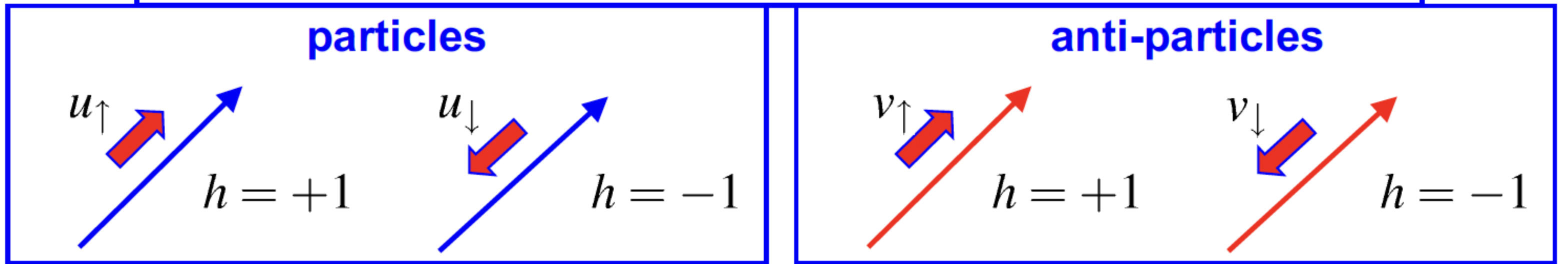
In the ultra-relativistic region, $E \gg m$, the 4 spinors can be approximated as

$$u_\uparrow\approx\sqrt{E}\begin{pmatrix}c\\ se^{i\phi}\\ c\\ se^{i\phi}\end{pmatrix},\ u_\downarrow\approx\sqrt{E}\begin{pmatrix}-s\\ ce^{i\phi}\\ s\\ -ce^{i\phi}\end{pmatrix},\ v_\uparrow\approx\sqrt{E}\begin{pmatrix}s\\ -ce^{i\phi}\\ -s\\ ce^{i\phi}\end{pmatrix}\ \text{and}\ v_\downarrow\approx\sqrt{E}\begin{pmatrix}c\\ se^{i\phi}\\ c\\ se^{i\phi}\end{pmatrix}.$$

Helicity Eigenstates

$$u_\uparrow = N \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N \begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N \begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$



Intrinsic Parity of Dirac Particles

Charge conjugation, a discrete symmetry that transforms,
particle ⟺ antiparticle.

Another discrete symmetry is the parity transformation, which corresponds to spatial inversion through the origin:

$$x' = -x, \quad y' = -y, \quad z' = -z \quad \text{and} \quad t' = t.$$

QED and QCD (strong interactions) always conserve parity.

Parity transformation properties & form of Parity operator

ψ is a solution of the Dirac equation; $$i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0\frac{\partial\psi}{\partial t}$$

ψ' is the parity transformation of ψ, $\psi \to \psi' = \hat{P}\psi.$

ψ' also satisfies Dirac $$i\gamma^1\frac{\partial\psi'}{\partial x'} + i\gamma^2\frac{\partial\psi'}{\partial y'} + i\gamma^3\frac{\partial\psi'}{\partial z'} - m\psi' = -i\gamma^0\frac{\partial\psi'}{\partial t'}$$

If you invert twice one Dirac solution you return to the initial solution $\psi' = \hat{P}\psi \quad \Rightarrow \quad \hat{P}\psi' = \psi.$

This means that $\hat{P}^2 = I$

Intrinsic Parity

Insert $\psi \to \psi' = \hat{P}\psi.$ into $i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial \psi}{\partial t}$

Change of reference system,
Dirac equation is always valid

$$i\gamma^1 \hat{P} \frac{\partial \psi'}{\partial x} + i\gamma^2 \hat{P} \frac{\partial \psi'}{\partial y} + i\gamma^3 \hat{P} \frac{\partial \psi'}{\partial z} - m\hat{P}\psi' = -i\gamma^0 \hat{P} \frac{\partial \psi'}{\partial t}$$

The derivatives in the Parity transformed system introduce a minus signs for all the space-like coordinates

$$x' = -x, \quad y' = -y, \quad z' = -z \quad \text{and} \quad t' = t.$$

Multiply all terms by γ^0

$$-i\gamma^0\gamma^1 \hat{P} \frac{\partial \psi'}{\partial x'} - i\gamma^0\gamma^2 \hat{P} \frac{\partial \psi'}{\partial y'} - i\gamma^0\gamma^3 \hat{P} \frac{\partial \psi'}{\partial z'} - m\gamma^0 \hat{P}\psi' = -i\gamma^0\gamma^0 \hat{P} \frac{\partial \psi'}{\partial t'}$$

$$\gamma^0\gamma^k = -\gamma^k\gamma^0$$

$$i\gamma^1\gamma^0 \hat{P} \frac{\partial \psi'}{\partial x'} + i\gamma^2\gamma^0 \hat{P} \frac{\partial \psi'}{\partial y'} + i\gamma^3\gamma^0 \hat{P} \frac{\partial \psi'}{\partial z'} - m\gamma^0 \hat{P}\psi' = -i\gamma^0\gamma^0 \hat{P} \frac{\partial \psi'}{\partial t'}$$

And compare with the starting equation

Explicit form of $\hat{P}$

$$i\gamma^1\hat{P}\frac{\partial\psi'}{\partial x} + i\gamma^2\hat{P}\frac{\partial\psi'}{\partial y} + i\gamma^3\hat{P}\frac{\partial\psi'}{\partial z} - m\hat{P}\psi' = -i\gamma^0\hat{P}\frac{\partial\psi'}{\partial t}$$

Initial equation with P-inverted coordinates

$$i\gamma^1\gamma^0\hat{P}\frac{\partial\psi'}{\partial x'} + i\gamma^2\gamma^0\hat{P}\frac{\partial\psi'}{\partial y'} + i\gamma^3\gamma^0\hat{P}\frac{\partial\psi'}{\partial z'} - m\gamma^0\hat{P}\psi' = -i\gamma^0\gamma^0\hat{P}\frac{\partial\psi'}{\partial t'}$$

Final equation

If we want the two equations to be the same

$$\gamma^0\hat{P} \propto I.$$

Since $\hat{P}^2 = I$ ⇒ $\hat{P} = +\gamma^0$ or $\hat{P} = -\gamma^0$

Conventional choice

Effect of parity on a spinor at rest

$$\hat{P}u_1 = \gamma^0 u_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = +u_1$$

$$\hat{P}u_2 = +u_2,\ \hat{P}v_1 = -v_1 \text{ and } \hat{P}v_2 = -v_2.$$

Hence the intrinsic parity of a fundamental spin-half particle is opposite to that of a fundamental spin-half antiparticle.